Assignment #5

Due: April 4, 2008
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Office hours: Thursday (4/3), 2-4 pm, or by appointment.

1. **Optical traps and scattering**

The AC Stark effect has been used to trap neutral atoms at the focus of one or more laser beams. In the following exercise, you will explore the proper laser power and wavelength needed to trap an ultracold atomic gas.

Let us again consider an alkali atom with resonance frequency ω_0 on the principal $nS \rightarrow nP$ transition. A sample of atoms in the ground state (nS) are exposed to monochromatic radiation of intensity I and frequency $\omega_L < \omega_0$ (red-detuned from resonance).

a) Calculate the AC Stark shift using both

i. the perturbative result for the dynamic polarizability $\alpha(\omega)$, and

ii. the nonperturbative approach in the rotating wave approximation (as treated in section 1.7 of the 2006 course notes and problem 2 from the first homework), but then approximated for small field strengths.

Calculate the ratio of results *(i)* / *(ii)* as a function of ω_L.

b) What is the likelihood of finding the atom in the excited state using both pictures *(i)* and *(ii)*? Neglect rapidly oscillating terms.

c) What is the photon scattering rate using *(i)* and *(ii)* as a function of ω_L? [Check: does your answer give the blue-sky formula (*scattering power* proportional to ω_L^4)?]

Finally, let us apply our formulae to the optical trapping of sodium. Atoms are to be trapped at the focus of a single laser beam. At the focal plane, the irradiance distribution has the form

$$I(r) = \frac{2P}{\pi w^2} \exp(-2r^2/w^2) \tag{1}$$

where $w = 6\,\mu$m is called the beam waist radius, r is the distance away from the center of the beam, and P is the laser power. Such a beam forms a trap for which the trap depth is given by the AC Stark shift at the maximum intensity $I(0)$.

d) We wish to trap a gas of sodium atoms with a temperature of about $1\,\mu$K. Thus, let us plan for a trap depth of $k_B \times 10\,\mu$K. Calculate the required power and the scattering rate for the following laser options:

1. **Yellow Option:** a detuning of 1.7 GHz (light at such a detuning is used for cooling sodium in the first place).
2. **Red Option:** the light of a He-Ne laser (633 nm). Incidentally, a run-of-the-mill laser pointer can put out 1 mW of power.

3. **Infrared Option:** the light of a diode laser at 985 nm.

You should use approach *(i)* which includes the counter-rotating term. For the Infrared Option, how much of a difference does this term make as a percentage of the total power needed and the scattering rate?

Hint: For alkali atoms, essentially all of the oscillator strength out of the ground state comes from the $nS \to nP$ transition (i.e. $f_{nP,nS} = 1$).

For reference:

Steven Chu *et al.* "Experimental observation of optically trapped atoms," PRL **57**, 314 (1986). First optical dipole trap for laser-cooled sodium atoms.

D.M. Stamper-Kurn *et al.* "Optical confinement of a Bose-Einstein condensate," PRL **80**, 2027 (1998). First optical dipole trap for Bose condensed sodium atoms.

2. **Atom-Field Interaction Hamiltonian**

In classical electromagnetic theory, a gauge transformation on the scalar potential $\phi(\mathbf{r},t)$ and vector potential $\mathbf{A}(\mathbf{r},t)$ is defined by the relations

$$\phi'(\mathbf{r},t) = \phi(\mathbf{r},t) - \frac{\partial}{\partial t} f(\mathbf{r},t) \quad (2)$$

$$\mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla f(\mathbf{r},t) \quad (3)$$

where $f(\mathbf{r},t)$ is an arbitrary differentiable function.

A gauge transformation is applied in quantum mechanics as a unitary transformation $O(\mathbf{r},t)$ (for which $OO^\dagger = 1$) which transforms states as

$$\psi'(\mathbf{r},t) = O(\mathbf{r},t)\psi(\mathbf{r},t). \quad (4)$$

Such a transformation transforms the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t) \quad (5)$$

into

$$i\hbar \frac{\partial}{\partial t}\psi'(\mathbf{r},t) = H'\psi'(\mathbf{r},t). \quad (6)$$

a) How is H' related to H and $O(\mathbf{r},t)$?

Now let us consider the interaction Hamiltonian (setting $c = 1$)

$$H^{int}(\mathbf{r},t) = \frac{1}{2m}[\mathbf{p} + q\mathbf{A}(\mathbf{r},t)]^2 - q\phi(\mathbf{r},t). \quad (7)$$

We introduce a gauge transformation by using the unitary operator

$$O(\mathbf{r},t) = e^{-iqf(\mathbf{r},t)/\hbar}. \quad (8)$$

b) How is H^{int} transformed by this gauge transformation? You should be able to identify the gauge transformation explicitly as defined in Eqs. (2) and (3).

Now, let us consider the case of no scalar potential ($\phi = 0$). The field is then described completely by the vector potential $\mathbf{A}(\mathbf{r},t)$. We choose a gauge transformation of the form

$$f(\mathbf{r},t) = -\mathbf{A}(t) \cdot \mathbf{r}. \quad (9)$$

In the spirit of the electric dipole approximation, let us neglect the spatial dependence of $\mathbf{A}(t)$ (meaning the wavelength of radiation is much larger than the size of our atom).

c) Find the transformed $H^{int'}$. Identify the electric field.

You may recall that in class, we adopted a different approach, where we expanded the interaction Hamiltonian as

$$\begin{aligned} H^{int} &= -\frac{q}{m}\mathbf{p}\cdot\mathbf{A} + \frac{q^2}{2m}|\mathbf{A}^2| \qquad (10) \\ &= H^{(1)} + H^{(2)} \qquad (11) \end{aligned}$$

We neglected the term $H^{(2)}$ because as a two-photon process, it contributes only at high intensities and for certain photon-scattering processes. As you have just shown, this step can be avoided using a gauge transformation. Also, notice that higher order terms can be obtained by considering the spatial dependence of the vector potential.

3. **Optical Transitions Driven by Blackbody Radiation**

Consider a cavity filled with blackbody radiation at a temparature T.

a) Find the average radiation energy density and the average number of photons per unit volume.

b) Find the rms E and B fields inside the cavity. What is E_{rms} inside this room?

c) Find the intensity of radiation escaping from a small hole in the wall of the cavity. (Just do the angular integral: you should get Stefan's law.)

d) The sun subtends 0.51 degrees from the earth, and the solar intensity at the earth is 1.4 kW/m^2. Find the blackbody temperature of the sun.

e) Do you have to worry about blackbody radiation when you trap a Bose-Einstein condensate in a magnetic trap? Any transition to another state will spin-flip the atoms and give the atoms recoil energy, causing the atom to be ejected from the trap. Assume that you want to ensure a trapping time of one minute. What is the maximum blackbody temperature one can tolerate assuming that the dominant electronic excitation has a transition wavelength of 590 nm and a lifetime of 16 ns?

$\int_0^\infty x^2(e^x-1)^{-1}dx = 2.404 \qquad \int_0^\infty x^3(e^x-1)^{-1} = \pi^4/15$

4. **RF Transition Lifetimes and RF Blackbody Transitions**

a) Estimate the lifetime of the hydrogen in the F=1 hyperfine level of the 1s state. The decay of F=1 to F=0 gives rise to the famous 21 cm line of radio astronomy. Assume that the matrix element is μ_B in your estimate.

b) A hydrogen Bose-Einstein condensate has been created in the F=1 state at MIT in the group of Dan Kleppner and Tom Greytak. At what rate does the blackbody radiation (at 4K and 300K) induce transitions to the F=0 state, which cannot be magnetically trapped?