8.421 Spring 2008 Homework Assignment #6 Solutions

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1.

We will use the same basis functions that were used for the resonant cavity. However, when the cavity is detuned from the atoms resonance, the two basis states do not have degenerate energies.

$$|I\rangle = |a, 0\rangle$$
$$|F\rangle = |b, 1\rangle$$

The Hamiltonian for the problem is made up of two parts. First, the diagonal terms are given by the energy of the basis states:

$$\langle I|\,\hat{\mathcal{H}}\,|I\rangle = \hbar\omega_I$$
$$\langle F|\,\hat{\mathcal{H}}\,|F\rangle = \hbar\omega_F$$
$$\delta \equiv \omega - \omega_{ab} = \omega_F - \omega_I$$

The second part of the Hamiltonian is the coupling of the two basis states. This coupling term shows up in the off-diagonal matrix elements:

$$\langle F|\,\hat{\mathcal{H}}\,|I\rangle = -i\hbar\frac{\Omega}{2}e^{i\delta t}$$

By an appropriate choice of the energy zero, we can write the Hamiltonian as:

$$\hat{\mathcal{H}} = \frac{\hbar}{2}\begin{pmatrix} -\omega_{FI} & i\Omega e^{-i\delta t} \\ -i\Omega e^{i\delta t} & \omega_{FI} \end{pmatrix}$$

This looks similar to problems we have worked before. We naturally try to eliminate the time dependence from the Hamiltonian. First we go to the interaction picture. The time dependent equations that we get are:

$$
\begin{aligned}
i\dot{c}_F &= \left(-i\frac{\Omega}{2}e^{i\delta t}\right)c_I \\
i\dot{c}_I &= \left(i\frac{\Omega}{2}e^{-i\delta t}\right)c_F
\end{aligned}
$$

Next we try the substitution:

$$
\begin{aligned}
c_F &= c_1 e^{\frac{i\delta t}{2}} \\
c_I &= c_2 e^{\frac{-i\delta t}{2}}
\end{aligned}
$$

and we get

$$
\begin{aligned}
i\dot{c}_1 &= \frac{\delta}{2}c_1 - i\frac{\Omega}{2}c_2 \\
i\dot{c}_2 &= i\frac{\Omega}{2}c_1 - \frac{\delta}{2}c_2
\end{aligned}
$$

So the new Hamiltonian is

$$
\hat{\mathcal{H}}_{inter} = \frac{\hbar}{2}\begin{pmatrix} \delta & -i\Omega \\ i\Omega & -\delta \end{pmatrix}
$$

This Hamiltonian is easily solved, and the resulting eigenstates are $E_{\pm} = \pm\frac{1}{2}\sqrt{\delta^2+\omega^2}$.

a)

The atom alone does not have stationary states, since it is interacting with the "environment" of the cavity. However, the atom - cavity system considered together is a closed system which evolves unitarily. The states $|I\rangle$ and $|F\rangle$ are not eigenstates. But by superposing these two states we have come up with two states which are eigenstates.

b)

To find the probability that the atom is in state $|a\rangle$, first we solve for the coefficients c_1 and c_2.

$$\left(\frac{E_\pm}{\hbar} - \frac{\delta}{2}\right) c_1 = i\frac{\Omega}{2} c_2$$
$$c_1 = \frac{i\Omega}{\pm\sqrt{\delta^2 + \Omega^2} - \delta} c_2$$

We know from normalization that ${c_1}^2 + {c_2}^2 = 1$. Now define θ such that:

$$\sin 2\theta = \frac{i\Omega}{\sqrt{\delta^2 + \Omega^2}}$$
$$\cos 2\theta = \frac{-\delta}{\sqrt{\delta^2 + \Omega^2}}$$

With this definition, we can write c_1 as:

$$c_1 = \frac{\sin 2\theta}{\pm 1 + \cos 2\theta} c_2$$

At $t = 0$, the system is in state $|a, 0\rangle = |I\rangle$, and the dressed states are

$$|+\rangle = \sin\theta\, |F\rangle + \cos\theta\, |I\rangle$$
$$|-\rangle = \cos\theta\, |F\rangle - \sin\theta\, |I\rangle$$

From this we get:

$$\Phi_0 = |I(0)\rangle = \cos\theta\, |+\rangle - \sin\theta\, |-\rangle$$
$$|I(t)\rangle = \cos\theta\, |+\rangle\, e^{\frac{-i\Omega' t}{2}} - \sin\theta\, |-\rangle\, e^{\frac{i\Omega' t}{2}}$$

where, as usual, $\Omega' = \sqrt{\delta^2 + \Omega^2}$.

$$P_a(t) = |\langle I(0)|I(t)\rangle|^2$$
$$= \frac{1}{2}\left[1 + \cos^2 2\theta + \left(\frac{\Omega}{\Omega'}\right)^2 \cos\Omega' t\right]$$

$$\boxed{P_a(t) = 1 - \left(\frac{\Omega}{\Omega'}\right)^2 \sin^2\frac{\Omega' t}{2}}$$

c)

$$\left\langle I(t)|\hat{\mathcal{H}}|I(t)\right\rangle = \cos^2\theta E_+ + \sin^2\theta E_-$$
$$= -\frac{1}{2}\delta$$

So the expectation value of the energy at all times is constant. If we start with the system in state $|I\rangle$, when we measure the energy at some time later, we may find the system in state $|F\rangle$, which has a different energy. This is not, however, a violation of energy conservation. Since $|I\rangle$ and $|F\rangle$ are not energy eigenstates, we expect their energy not to be well defined. When we measure the system it is forced out of the dressed state and collapses into one of the uncoupled basis states. When this happens, the system can exchange a tiny bit of energy with our measurement apparatus in order to conserve energy. This makes it possible to put the atom into the cavity in state $|a,0\rangle$, and later find it in state $|b,1\rangle$, even though that state has a different energy.

2.

The unsaturated rate is given by

$$R^u = 6\pi\bar{\lambda}^2\frac{I}{\hbar\omega}$$

A saturation parameter of one means that

$$\begin{aligned}
R^u &= \frac{\Gamma}{2} \\
&= 6\pi\bar{\lambda}^2\frac{I_{saturation}}{\hbar\omega} \\
I_{saturation} &= \frac{\hbar\omega\Gamma}{12\pi\bar{\lambda}^2} \\
&= \frac{\Gamma}{\lambda^3}\frac{\pi hc}{3} \\
&= \frac{1}{(16\cdot 10^{-9}sec)(589\cdot 10^{-9}meters)}\frac{\pi hc}{3} \\
&= 64\frac{watts}{meter}
\end{aligned}$$

$$\boxed{I_{saturation} = 6.4\frac{mW}{cm^2}}$$

3.

This can be done most easily in atomic units. Then the answer can be easily converted to Hz.

a)

$$\omega_{50} - \omega_{49} = (50)^{-3} atomicunits = 8 \cdot 10^{-6} atomicunits$$
$$= (8 \cdot 10^{-6}) 2cR = (8 \cdot 10^{-6})(6.58 \cdot 10^{15} Hz)$$

$\boxed{\omega_{50} - \omega_{49} = 52.6GHz}$

In a cubical cavity, $k = (k_x^2 + k_y^2 + k_z^2)^{\frac{1}{2}} = \frac{\pi}{L}(m^2 + n^2 + p^2)^{\frac{1}{2}}$. The lowest mode which satisfies the boundary conditions is $m = n = 1, p = 0$.

$$k = \sqrt{2}\frac{\pi}{L}$$
$$= \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$$
$$L = \frac{c}{\sqrt{2}\nu}$$

$\boxed{L = 4.03mm}$

b)

In atomic units

$$\Omega = z_{ab}\sqrt{\frac{8\pi\omega}{V}}$$
$$= n^2\sqrt{\frac{8\pi}{n^3 V}}$$
$$= \sqrt{\frac{8\pi n}{L^3}}$$
$$= \sqrt{\frac{50 \cdot 8\pi}{(\frac{0.403}{0.53 \cdot 10^{-8}})^3}}$$
$$= 5.35 \cdot 10^{-11} atomicunits$$

$\boxed{\Omega = 352kHz}$