

Homework Assignment #7

Physics 8.421, Spring 2008, Prof. W. Ketterle and Prof. V. Vuletic

Due Monday, Apr. 21, 2008

1. Convolution of Line Shapes

When two separate physical processes both contribute to the line shape, the resultant line shape is the convolution of the distributions. Say $D_1(\omega - \omega_0)$ and $D_2(\omega - \omega_0)$ are the normalized line shapes of the first and second type of process. The resultant line shape is then their convolution

$$D_R(\omega - \omega_0) = \int_{-\infty}^{\infty} d\omega' D_1(\omega - \omega') D_2(\omega' - \omega_0)$$

- (1 point) Consider that D_1 and D_2 are both Lorentzian functions with FWHMs (full widths at half-maximum) Γ_1 and Γ_2 , respectively. Show that D_R is also Lorentzian and find its FWHM. (Hint: use the Fourier convolution theorem)
- (1 point) Do the same if D_1 and D_2 are Gaussian and Γ_1 and Γ_2 are r.m.s. deviations.
- (1 point) Find the FWHM of a Gaussian with r.m.s. deviation $\Gamma/2$. Find the r.m.s. deviation of a Lorentzian with FWHM Γ .

2. Saturation of Atomic Transitions

In class we discussed excitation of atoms via weak radiation. In this limit the atom scatters incident radiation at a rate proportional to the light intensity, corresponding to a fixed cross-section.

We also discussed the excitation of atoms via strong radiation and showed that in this limit the atom performs Bloch oscillations between the ground and excited states. Since during these oscillations the mean excited state population is at most 1/2 and the excited state decays with rate Γ , the atom can scatter at most $\Gamma/2$ photons per unit time. To obtain a fixed scattering rate, as the radiation intensity increases, the photon-scattering cross-section decreases, becoming very low at high light intensities.

This problem will motivate this saturation of atomic transitions by considering broadband excitation. Surprisingly, the obtained results can be exactly extended to narrowband transitions.

- (1 point) In the case of broadband excitation, the atom dynamics is correctly described by the Einstein rate equations. Consider a two-state atom with $R_{ge} = R_{eg}$ the stimulated absorption/emission rate and $A = \Gamma$ the spontaneous emission rate. Define the saturation parameter s as $s = 2R_{ge}/\Gamma$. Show that in equilibrium the ratio of the excited state to the ground state populations is $N_b/N_a = s/(s+2)$.
- (1 point) Express the equilibrium spontaneous emission rate per atom AN_b in terms of Γ and s . Show that the cross-section for photon absorption bleaches out as $\sigma(s) = \sigma(s=0)/(1+s)$.
- (1 point) Find the energy density $\langle w \rangle_{SAT}$ per unit frequency corresponding to $s = 1$. Explain why $\langle w \rangle$ is independent of the atomic dipole matrix element $\langle g | er | e \rangle$.
- (2 points) Use the relationship between Einstein's A and B coefficients to obtain an expression for $\langle w \rangle_{SAT}$ independent of the atomic dipole. For $s = 1$, what is the mean occupation number n per photon mode?
- (1 point) Suppose that the light is provided by a laser beam of intensity I_0 and Lorentzian lineshape centered at the atomic transition frequency ω_0 and of FWHM $\Gamma' \gg \Gamma$. What is the energy density of this beam per frequency interval at ω_0 ? What beam intensity I_s corresponds to $s = 1$?

f) (2 points) Let ω_R be the Rabi frequency corresponding to a monochromatic beam with the same intensity I_0 as the broadband beam. Use the derivation from Lecture 12 to show that the stimulated broadband absorption rate can be written as $R = \omega_R^2/\Gamma'$. Use the fact that R is also given by Einstein's A and B coefficients to express ω_R^2 in terms of I_0 and Γ . What is ω_R^2 corresponding to $s = 1$?

g) (1 point) If you set $\Gamma' = \Gamma$, you get exactly the saturation intensity of a monochromatic laser beam and the Rabi frequency at saturation. Argue why.

3. Optical Bloch Equations

The previous problem obtained the correct result for the atomic saturation intensity for narrowband radiation without considering the real atom dynamics. In this problem we will revisit this result using a more detailed approach.

Remember that for weak narrowband excitation, the atomic dynamics is described by a rate process while for strong excitation the correct description is in terms of the coherent evolution of the atomic wave function. Since saturation occurs between these two regimes, we need a technique which combines the two descriptions. This is provided by density matrices.

a) (1 point) Consider a two-level system driven at a large detuning $|\delta| \gg \Gamma, \omega_R$. The system is in the ground state $|g\rangle$ at time $t = 0$. If we neglect decay of the excited state, what is the excited state fraction $\rho_{ee}(t)$ given by undamped Rabi oscillations? What do you expect will happen if a weak damping term is added to account for spontaneous emission? Guess the result for $\rho_{ee}(t)$ in the limit $t \rightarrow \infty$.

b) (1 point) We will now carefully consider the effect of spontaneous emission, including the resonant case. In class we argued that the coupling of an excited atomic state to a continuum of radiation modes gives rise to an imaginary part $i\Gamma/2$ of the excited state energy, corresponding to decay of the excited state with rate Γ . If we start the atom in the excited state $|e\rangle$ at $t = 0$, the normalized state of the atom + radiation field at time t will be:

$$|\psi(t)\rangle = e^{-i\omega_e t - \Gamma t/2} |e\rangle \sum_{\omega, k} |\omega, k, 0\rangle + \sum_{\omega, k} A(\omega, k) |g\rangle |\omega, k, 1\rangle$$

Suppose now we start the atom in the state $\alpha |g\rangle + \beta |e\rangle$. Show that the reduced density matrix of the atom $\rho(t)$ at time t , obtained by taking a trace of the total density matrix $\rho_S(t) = |\psi(t)\rangle \langle \psi(t)|$ over the modes $|\omega, k, n\rangle$ of the radiation field, takes the form:

$$\rho(t) = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix} = \begin{pmatrix} 1 - |\beta|^2 e^{-\Gamma t} & \bar{\alpha}\beta e^{i(\omega_g - \omega_e)t - \Gamma t/2} \\ \bar{\beta}\alpha e^{-i(\omega_g - \omega_e)t - \Gamma t/2} & |\beta|^2 e^{-\Gamma t} \end{pmatrix}$$

The diagonal elements of the density matrix decay twice as fast as the off-diagonal terms. Why?

c) (1 point) In the absence of spontaneous emission, the evolution of the atomic density matrix is given by $\dot{\rho} = \frac{1}{i\hbar} [H_0, \rho]$ with

$$H_0 = \begin{pmatrix} \hbar\omega_g & 0 \\ 0 & \hbar\omega_e \end{pmatrix}$$

the unperturbed atomic Hamiltonian. Show that when spontaneous emission is included as above, the evolution of the atomic density matrix can be written as:

$$\dot{\rho} = \frac{1}{i\hbar} [H_0, \rho] - \Gamma \begin{pmatrix} -\rho_{ee} & \rho_{ge}/2 \\ \rho_{eg}/2 & \rho_{ee} \end{pmatrix}$$

This master equation combines the coherent evolution of the density matrix under the atom Hamiltonian H with decoherence due to spontaneous emission.

d) (1 point) We now add the drive due to a laser beam. If we refer all energies to the midpoint between the two atomic levels, we may write the perturbed Hamiltonian in the rotating wave approximation as:

$$H = \frac{\hbar}{2} \begin{pmatrix} -\omega_0 & \omega_R e^{i\omega t} \\ \omega_R e^{-i\omega t} & +\omega_0 \end{pmatrix}$$

Solve for the time-independent form of the master equation by making the substitution $\rho_{ge} \rightarrow \rho_{ge} e^{+i\omega t}$ and $\rho_{eg} \rightarrow \rho_{eg} e^{-i\omega t}$ to get:

$$\begin{aligned} \dot{\rho}_{gg} &= +i\frac{\omega_R}{2} (\rho_{ge} - \rho_{eg}) + \Gamma\rho_{ee} \\ \dot{\rho}_{ee} &= -i\frac{\omega_R}{2} (\rho_{ge} - \rho_{eg}) - \Gamma\rho_{ee} \\ \dot{\rho}_{ge} &= \left(-i\delta - \frac{\Gamma}{2}\right) \rho_{ge} + i\frac{\omega_R}{2} (\rho_{gg} - \rho_{ee}) \\ \dot{\rho}_{eg} &= \left(+i\delta - \frac{\Gamma}{2}\right) \rho_{eg} - i\frac{\omega_R}{2} (\rho_{gg} - \rho_{ee}) \end{aligned}$$

with $\delta = \omega - \omega_0$ the laser beam detuning.

e) (1 point) Show that in the limit $|\delta| \gg \Gamma$, ω_R and with the atom initially in the ground state, the master equation gives:

$$\rho_{ee} = \frac{\omega_R^2}{4\delta^2} \left(1 + e^{-\Gamma t} - 2e^{-\Gamma t/2} \cos \delta t\right)$$

What does this reduce to in the limit $t \rightarrow \infty$?

f) (2 points) The excited state population ρ_{ee} decays with rate Γ to the ground state via spontaneous emission, corresponding to photon scattering rate $R_{sc} = \Gamma\rho_{ee}$. Show that the steady-state photon scattering rate for arbitrary δ , Γ and ω_R can be written as:

$$R_{sc} = \Gamma\rho_{ee} = \frac{\Gamma}{2} \frac{s}{1 + s + (\delta/(\Gamma/2))^2}$$

with $s = 2\omega_R^2/\Gamma^2$ the saturation parameter obtained in Problem 2.

g) (1 point) Your answer for a) may differ from the correct answer from d). In general, for a driven two-level system in contact with a reservoir, the equilibrium excited state population will depend on the exact model for decoherence. It can be shown that the most general such model corresponds to

$$\dot{\rho} = \frac{1}{i\hbar} [H, \rho] - \begin{pmatrix} -\rho_{ee}/T_1 & \rho_{ge}/T_2 \\ \rho_{eg}/T_2 & \rho_{ee}/T_1 \end{pmatrix}$$

Find the equilibrium excited state population in this general case, commonly encountered in spin NMR. Here the T_1 and T_2 times correspond to the two eponymous decoherence mechanisms.

For questions or assistance with this assignment contact:

Marko Cetina
mcetina@mit.edu
617-452-5092

Office hours will be on Thursday, 7:00pm in 26-225, or by e-mail request.