# 8.421 - PS7 SOLUTIONS

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## Problem 1

The resultant line shape is

$$\begin{aligned} D_R(\omega - \omega_0) &= \int_{-\infty}^{\infty} d\omega' D_1(\omega - \omega') D_2(\omega' - \omega_0) \\ D_R(\delta) &= \int_{-\infty}^{\infty} d\delta' D_1(\delta - \delta') D_2(\delta') \end{aligned}$$

where $\delta = \omega - \omega_0$, and $\delta' = \omega' - \omega_0$ is the dummy variable.

Shorthand notation for this convolution is $D_R(\delta) = D_1(\delta) \otimes D_1(\delta)$. The convolution theorem states that the convolution of two functions is the inverse Fourier transform of the product of their fourier transforms. Thus we have

$$D_1(\delta) \otimes D_2(\delta) \Longleftrightarrow d_1(t) d_2(t)$$

where $d(t) = \int_{-\infty}^{\infty} e^{-i\delta t} D(\delta) d\delta$ and the double arrow indicates that the two sides are Fourier transforms of each other.

a) If $D(\delta) = \frac{1}{\pi} \frac{\Gamma/2}{\delta^2 + (\Gamma/2)^2}$, its transform is $d(t) = e^{-\Gamma|t|/2}$. (You can show this by doing a contour integral or just find it in a table of Fourier transformations.) So,

$$\begin{aligned} D_R(\delta) &= F^{-1}[d_1(t) d(_2(t)] \\ &= F^{-1}[e^{-(\Gamma_1 + \Gamma_2)|t|/2}] \\ &= \frac{1}{\pi} \frac{(\Gamma_1 + \Gamma_2)/2}{\delta^2 + [(\Gamma_1 + \Gamma_2)/2]^2} \end{aligned}$$

Also Lorentzian with FWHM= $\Gamma_1 + \Gamma_2$.

b) If $D(\delta) = \frac{1}{\sqrt{2\pi\Gamma^2}} e^{-\delta^2/2\Gamma^2}$ (r.m.s. deviation $\sqrt{< \delta^2 >} = \Gamma$), its transform is $d(t) = e^{-\Gamma^2 t^2/2}$. The resultant line shape is

$$\begin{aligned} D_R(\delta) &= F^{-1}[d_1(t) d(_2(t)] \\ &= F^{-1}[e^{-(\Gamma_1^2 + \Gamma_2^2)t^2/2}] \\ &= \frac{1}{\sqrt{2\pi(\Gamma_1^2 + \Gamma_2^2)}} e^{-\delta^2/2(\Gamma_1^2 + \Gamma_2^2)} \end{aligned}$$

Also Gaussian with r.m.s. deviation= $(\Gamma_1^2 + \Gamma_2^2)\,(1/2)$.

c) This question is playing with the different definitions of linewidth. A Gaussian with r.m.s deviation $\Gamma/2$, $D(\delta) = \sqrt{\frac{2}{\pi\Gamma^2}}e^{-2\delta^2/\Gamma^2}$ has FWHM of $2\delta_{1/2}$, s.t. $D(\delta_{1/2}) = D(0)/2$. Then FWHM=$\Gamma\sqrt{2ln2} = 1.18\Gamma$.

Since $\int_{-\infty}^{\infty} d\delta \frac{\delta^2}{\delta^2+(\Gamma/2)^2}$ is diverging, the r.m.s. deviation for Lorentizian distribution is undefined.

## Problem 2

a) With $N_e$ and $N_g$ the ground and excited state populations, the Einstein rate equations can be written as:

$$\begin{aligned}\dot{N}_b &= -N_b\left(R_{eg}+\Gamma\right)+R_{ge}N_a \\ \dot{N}_a &= +N_b\left(R_{eg}+\Gamma\right)-R_{ge}N_a\end{aligned}$$

with $\Gamma = A$, $R_{eg} = B_{eg}\langle w\rangle$, $R_{ge} = B_{ge}\langle w\rangle$ and $\langle w\rangle$ the energy density per frequency interval of the driving field. In equilibrium, $\dot{N}_a = \dot{N}_b = 0$ so

$$N_b\left(R_{eg}+\Gamma\right) = R_{ge}N_a$$

and

$$\frac{N_b}{N_a} = \frac{R_{ge}}{R_{eg}+\Gamma} = \frac{2R_{ge}/\Gamma}{2R_{eg}/\Gamma+2} = \frac{s}{s+2}$$

b) The total number of atoms is constant: $N_a + N_b = N$ while from the previous problem, in equilibrium, $N_b/N_a = s/\left(s+2\right)$, so:

$$N_b = \frac{N}{2}\frac{s}{s+1}$$

and the spontaneous emission rate per atom is:

$$\frac{AN_b}{N} = \frac{\Gamma}{2}\frac{s}{s+1}$$

The atoms absorb light from the incident field by stimulated absorption and return it to the incident field via stimulated emission. The total power is conserved so the difference between these two is equal to the spontaneous emission power $W$:

$$W = R_{ge}N_a - R_{eg}N_b = AN_b$$

By definition of the photon scattering cross-section $\sigma$, the scattered power can also be written as $W = \sigma \times I_0$ with $I_0$ the incident field intensity $I_0$. Since $I_0 \sim \langle w\rangle$:

$$\sigma = \frac{\hbar\omega AN_b}{I_0} \sim \frac{AN_b}{\langle w\rangle} = \frac{AN_b}{R_{ge}/B_{ge}} \sim \frac{N_b}{2R_{ge}/\Gamma} = \frac{N_b}{s}$$

and

$$\sigma\left(s\right) \sim \frac{N}{2}\frac{s}{s+1}/s \sim \frac{1}{s+1}$$

which bleaches out as $\sigma(s) = \sigma(s=0)/(s+1)$.

c)

$$1 = s = \frac{2R_{ge}}{\Gamma} = \frac{2B_{ge}\langle w\rangle}{A} \Rightarrow \langle w\rangle = \langle w\rangle_{SAT} = \frac{1}{2}\frac{A}{B_{ge}}$$

The relationship between Einstein's A and B coefficients and hence the saturation energy density is independent of the details of the two-level atom under considerations, including its dipole matrix element. The reason for this is that both coefficients describe rate processes driven by the electromagnetic field – the former the spontaneous emission driven by the vacuum field and the latter the stimulated absorption/emission driven by the radiation source. The ratio between these two rates then depends only on the source energy density relative to the energy density of the vacuum field.

d) From the Einstein A and B coefficient derivation,

$$A = \rho(\omega)\frac{\hbar\omega}{V}B_{eg}$$

where $V$ the volume of the radiation box and $\rho(\omega)$ is the density of states in 3D including both light polarizations:

$$\rho(\omega) = 2 \times \left(\frac{V^{1/3}}{2\pi}\right)^3 4\pi(\omega/c)^2/c = \frac{\omega^2}{\pi^2 c^3}V$$

For two-level atom $R_{eg} = R_{ge}$ and $B_{eg} = B_{ge}$ so:

$$\langle w\rangle_{SAT} = \frac{1}{2}\frac{A}{B_{ge}} = \frac{1}{2}\left(\frac{B_{eg}}{B_{ge}}\right)\frac{A}{B_{eg}} = \frac{1}{2}\frac{\hbar\omega^3}{\pi^2 c^3}$$

In terms of the mean number of photons per mode $\langle n(\omega)\rangle$:

$$\langle w(\omega)\rangle = \frac{\hbar\omega}{V}\rho(\omega)\langle n(\omega)\rangle$$

For $s = 1$, corresponding to stimulated absorption rate one half of the spontaneous emission rate:

$$B_{ge}\langle w(\omega)\rangle = B_{ge}\frac{\hbar\omega}{V}\rho(\omega)\langle n(\omega)\rangle = A/2 = \rho(\omega)\frac{\hbar\omega}{V}B_{eg}/2$$

so

$$\langle n(\omega)\rangle_{SAT} = \frac{1}{2}\frac{B_{eg}}{B_{ge}} = \frac{1}{2}$$

In other words, the stimulated emission rate due to vacuum fluctuations corresponds to one photon per mode of the electromagnetic field.

*A note on polarization:*

This problem only considers two-state atoms. A real atom will have degeneracies in the ground and excited states, in which case Einstein's A and B coefficients give:

$$R_{eg} = B_{eg}\langle w(\omega)\rangle \quad ; \quad R_{ge} = B_{ge}\langle w(\omega)\rangle$$
$$g_g R_{ge} = g_e R_{eg} \quad ; \quad A = \rho(\omega)\frac{\hbar\omega}{V}B_{eg}$$

In the simplest case of a transition between $J = 0$ and $J = 1$ levels in an $I = 0$ atom, the net effect is a 3x increase in the stimulated absorption rate over the stiumulated and spontaneous emission rates. The reason for this is that an atom in the ground state can absorb all light polarizations while an atom in a given excited $m$-state has to emit one given polarization to return to the ground state.

If the atom is driven by a polarized source, it will be similarily polarized and the stimulated emission rate due to the source will still be equal to the (unchanged) stimulated absorption rate. The saturation parameter will still be $s = 2R_{ge}/\Gamma$ but with $\Gamma$ 1/3rd of its original value, corresponding to 1/3rd of the vacuum modes having the right polarization for spontaneous emission. Consequently, the saturation intensity will be 1/3rd of the scalar value:

$$\langle w \rangle_{SAT} = \frac{1}{3} \times \frac{1}{2} \frac{\hbar\omega^3}{\pi^2 c^3}$$

while the mean number of photons per mode of the *polarized* driving field will be 1/6.

e) The energy density of a beam with intensity $I_0$ will be $I_0/c$. Meanwhile, for a Lorentzian lineshape of FWHM $\Gamma'$, the power density will be:

$$I(\omega) = \frac{I_0}{\pi} \frac{(\Gamma'/2)}{(\omega - \omega_0)^2 + (\Gamma'/2)^2}$$

Indeed, then:

$$\begin{aligned}
I(\omega_0) &= \frac{I_0}{\pi} \frac{1}{(\Gamma'/2)} \\
I(\omega_0 + \Gamma'/2) &= \frac{I_0}{\pi} \frac{(\Gamma'/2)}{(\Gamma'/2)^2 + (\Gamma'/2)^2} = \frac{1}{2} I(\omega_0)/2 \\
\int_{-\infty}^{\infty} I(\omega)\, d\omega &= I_0 \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{(1/2)}{\omega^2 + (1/2)^2} d\omega \\
&= \frac{I_0}{\pi} 2\pi i \operatorname{Res}\left[\frac{(1/2)}{\omega^2 + (1/2)^2}\right]_{\omega = i/2} \\
&= \frac{I_0}{\pi} 2\pi i \left(-\frac{i}{2}\right) = I_0
\end{aligned}$$

To reach $s = 1$ at $\omega = \omega_0$:

$$\frac{1}{c} \frac{I_0}{\pi} \frac{1}{\Gamma'/2} = \frac{1}{2} \frac{A}{B} = \frac{1}{2} \frac{\hbar\omega^3}{\pi^2 c^3}$$

or

$$I_0 = I_s = \frac{\hbar\omega^3}{4\pi c^2} \times \Gamma' = \frac{\pi}{\lambda^2} \times \hbar\omega \times \Gamma'$$

If we include the effect of polarization as in part d), $I_s = \hbar\omega^3\Gamma'/12\pi c^2$.

f) From Lecture 12, by first-order perturbation theory, in the rotating-wave approximation, for a driving field at detuning $\Delta$, the excited state amplitude will be:

$$c_e(t) = \frac{1}{i}\int_0^t d\tau \, \frac{\omega_R}{2} e^{-i\Delta\tau} = \frac{\omega_R}{\Delta}\frac{e^{-i\Delta t/2}}{i}\sin\left(\frac{\Delta}{2}t\right)$$

Under broadband incoherent drive field, the various frequency components of the driving field will contribute incoherently to the excited state amplitudes. Consequently, we add the corresponding populations to find the excited state probability:

$$\begin{aligned} p_e(t) &= \int_{-\infty}^{\infty} \tilde{\omega}_R^2(\Delta)\frac{\sin^2(\Delta t/2)}{\Delta^2} d\Delta \\ &= \frac{t}{2}\int_{-\infty}^{\infty} \tilde{\omega}_R^2(\Delta)\frac{\sin^2(\Delta t/2)}{(\Delta t/2)^2} d(\Delta t/2) \\ &= \frac{\pi}{2}\tilde{\omega}_R^2(0)\times t \end{aligned}$$

where $\tilde{\omega}_R^2(\Delta)$ is the power spectral density of the Rabi frequency. Since $\omega_R^2$ is proportional to the beam intensity $I_0$, $\tilde{\omega}_R^2$ will be proportional to the power spectral density $I(\omega)$ of the beam intensity. Since for a Lorentzian $I(\omega)$ with FWHM $\Gamma'$ from the previous problem $I(\Delta = 0) = \frac{I_0}{\pi}\frac{1}{\Gamma'/2}$, we also have $\tilde{\omega}_R^2(0) = \frac{\omega_R^2}{\pi}\frac{1}{\Gamma'/2}$ and so:

$$p_e(t) = \frac{\pi}{2}\frac{\omega_R^2}{\pi}\frac{1}{\Gamma'/2}t = \frac{\omega_R^2}{\Gamma'}t$$

corresponding to an excitation rate of $R_{ge} = \omega_R^2/\Gamma'$. Meanwhile, from the Einstein A and B coefficients:

$$R_{ge} = B_{ge}\frac{I(\omega)}{c} = \frac{B_{ge}}{B_{eg}}\Gamma/\left(\rho(\omega)\frac{\hbar\omega}{V}\right)\frac{I_0}{c\pi}\frac{1}{\Gamma'/2}$$

so, in 3D, for a two-level atom:

$$\omega_R^2 = \frac{\pi^2 c^3}{\hbar\omega^3}\frac{2}{\pi}\frac{I_0}{c}\Gamma = \frac{2\pi c^2}{\hbar\omega^3}\frac{I_0}{c}\Gamma$$

If we include polarization, $\Gamma$ decreases 3 times so

$$\omega_R^2 = \frac{6\pi c^2}{\hbar\omega^3}\times I_0\Gamma$$

At $s = 1$, $2R_{ge} = \Gamma$ so $\omega_R^2 = \Gamma'\Gamma/2$.

g) Spontaneous emission i.e. coupling to the continuum of radiation modes broadens the excited state to a linewidth $\Gamma$. Consequently, the atom cannot really 'see' the true linewidth of a laser with linewidth smaller than $\Gamma$, perceiving it instead as a broad source of linewidth $\Gamma$. Then, as seen by the atom $\Gamma' = \Gamma$ and we get the above saturation result.

## Problem 3

a) The excited state probability for a two level system driven with arbitrary detuning $\delta$ and field strength $\omega_R$ is given by the Rabi solution:

$$\rho_{ee}(t) = \frac{\omega_R^2}{2}\sin^2\left(\frac{\omega_R' t}{2}\right)$$

where $\omega_R' \equiv \sqrt{\omega_R^2 + \delta^2}$. For a large detuning, $\omega_R' \to \delta$ and

$$\rho_{ee}(t) = \frac{\omega_R^2}{\delta^2} \sin^2\left(\frac{\delta\, t}{2}\right)$$

You might expect that with spontaneous emission, the $\sin^2(\delta t/2)$ oscillation in the excited state population would damp out to an average value of 1/2 and $\rho_{ee}(t \to \infty) = \omega_R^2/2\delta^2$.

b) Denote the vacuum state of the field as $|0\rangle$ and the state with one photon in the $(\omega, k)$ mode as $|\omega, k, 1\rangle$. If the atom is started in the ground state, under the rotating wave approximation it will stay there. Hence, in this case:

$$|\psi(t)\rangle = e^{-i\omega_g t} |g\rangle |0\rangle$$

By linearity, if we start the atom in $\alpha |g\rangle + \beta |e\rangle$, we will have:

$$\begin{aligned} |\psi(t)\rangle &= \alpha e^{-i\omega_g t} |g\rangle |0\rangle + \beta e^{-i\omega_e t - \Gamma t/2} |e\rangle |0\rangle + \beta \sum_{\omega,k} A(\omega, k, t) |g\rangle |\omega, k, 1\rangle \\ &= |\phi(t)\rangle |0\rangle + \beta \sum_{\omega,k} A(\omega, k, t) |g\rangle |\omega, k, 1\rangle \end{aligned}$$

where we introduce an unnormalized atom state $|\phi(t)\rangle = e^{-i\omega_g t}\alpha |g\rangle + e^{-i\omega_e t - \Gamma t/2}\beta |e\rangle$ corresponding to the coherent part of the atomic evolution. The decay of the norm of this state corresponds to the transfer of population into the ground state via spontaneous emission. Then:

$$\begin{aligned} \rho(t) &= |\psi(t)\rangle \langle \psi(t)| = |\phi(t)\rangle |0\rangle \langle 0| \langle \phi(t)| + |\beta|^2 \sum_{\omega,k} |A(\omega, k, t)|^2 |g\rangle |\omega, k, 1\rangle \langle \omega, k, 1| \langle g| \\ &+ \left( \sum_{\omega,k} \bar{A}(\omega, k) \bar{\beta} |\phi(t)\rangle |0\rangle \langle \omega, k, 1| \langle g| + c.c. \right) \end{aligned}$$

Since $\mathrm{Tr}(|0\rangle\langle 0|) = \mathrm{Tr}(|\omega, k, 1\rangle \langle \omega, k, 1|)$ and $\mathrm{Tr}(|0\rangle \langle \omega, k, 1|) = 0$, the last term drops out and:

$$\rho_{atom}(t) = \mathrm{Tr}_{radiation}\rho(t) = |\phi\rangle\langle\phi| + |\beta|^2 \left( \sum_{\omega,k} |A(\omega, k, t)|^2 \right) |g\rangle \langle g|$$

The total system wavefunction given in the problem is normalized so:

$$\left| e^{-\Gamma t/2} \right|^2 + \sum_{\omega,k} |A(\omega, k)|^2 = 1$$

and:

$$\begin{aligned}
\rho_{atom}(t) &= |\phi(t)\rangle\langle\phi(t)| + \left(1 - |\phi(t)|^2\right)|g\rangle\langle g| \\
&= \begin{pmatrix} |\alpha|^2 & \bar{\beta}\alpha\, e^{-i(\omega_g-\omega_e)t-\Gamma t/2} \\ \bar{\alpha}\beta\, e^{i(\omega_g-\omega_e)t-\Gamma t/2} & |\beta|^2 e^{-\Gamma t} \end{pmatrix} \\
&+ \begin{pmatrix} 1 - |\alpha|^2 - |\beta|^2 e^{-\Gamma t} & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 - |\beta|^2 e^{-\Gamma t} & \bar{\beta}\alpha\, e^{-i(\omega_g-\omega_e)t-\Gamma t/2} \\ \bar{\alpha}\beta\, e^{i(\omega_g-\omega_e)t-\Gamma t/2} & |\beta|^2 e^{-\Gamma t} \end{pmatrix}
\end{aligned}$$

Note that for this $\rho$, $\mathrm{Tr}\left(\rho^2\right) < 1$ i.e. this density matrix does not correspond to a pure state. The reason for this is that spontaneous emission admixes an incoherent ground state mixture into the atomic ensemble which is not described by the wavefunction $|\phi\rangle$.

Also note that the last expression for $\rho$ given in the problem was mistakenly transposed.

Both the *ee* and the off-diagonal terms (coherences) are nonzero only because of the excited amplitude in $|\phi(t)\rangle$. Due to coupling to the radiation modes, this amplitude decays as $e^{-\Gamma t/2}$. Since the coherences are proportional to this excited state amplitude and the excited state population to its square, the population decays twice as fast as the coherences.

c) With $\rho(t)$ the reduced atomic density matrix from part b), by direct time differentiation we get:

$$\begin{aligned}
\dot{\rho} &= -\Gamma \begin{pmatrix} -|\beta|^2 e^{-\Gamma t} & \rho_{ge}/2 \\ \rho_{eg}/2 & |\beta|^2 e^{-\Gamma t} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & (-i)(\omega_g - \omega_e)\rho_{eg} \\ i(\omega_g - \omega_e)\rho_{ge} & 0 \end{pmatrix}
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\frac{1}{i\hbar}[H_0, \rho] &= \frac{1}{i\hbar}\left[\begin{pmatrix} \hbar\omega_g & 0 \\ 0 & \hbar\omega_e \end{pmatrix}, \begin{pmatrix} 0 & \rho_{ge} \\ \rho_{eg} & 0 \end{pmatrix}\right] \\
&= \frac{1}{i}\left[\begin{pmatrix} 0 & \omega_g\rho_{ge} \\ \omega_e\rho_{eg} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \omega_e\rho_{ge} \\ \omega_g\rho_{eg} & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & (-i)(\omega_g - \omega_e)\rho_{ge} \\ (-i)(\omega_e - \omega_g)\rho_{eg} & 0 \end{pmatrix}
\end{aligned}$$

Since $\rho_{ee} = |\beta|^2 e^{-\Gamma t}$ we see that for arbitrary $\alpha$ and $\beta$ i.e. for an arbitrary initial pure state of the atom, the evolution of $\rho$ satisfies:

$$\dot{\rho} = \frac{1}{i\hbar}[H_0, \rho] - \Gamma \begin{pmatrix} -\rho_{ee} & \rho_{ge}/2 \\ \rho_{eg}/2 & \rho_{ee} \end{pmatrix}$$

Since the density matrix description is linear in statistical mixtures, this equation also extends to arbitrary mixtures of states at $t = 0$. Conversely, for an arbitrary initial condition $\rho(t = 0)$, the above equation specifies a unique time evolution of the atomic density matrix.

d) Making the substitution $\rho_{ge} \to \rho_{ge} e^{+i\omega t}$ and $\rho_{eg} \to \rho_{eg} e^{-i\omega t}$ we may write:

$$\begin{aligned} 2H/\hbar &= -\omega_0 \sigma_z + \omega_R \sigma_+ e^{i\omega t} + \omega_R \sigma_- e^{-i\omega t} \\ \rho &= \rho_{ge} e^{i\omega t} \sigma_+ + \rho_{eg} \sigma_- e^{-i\omega t} + \frac{\rho_{gg} - \rho_{ee}}{2} \sigma_z \end{aligned}$$

with

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad ; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the Pauli $\sigma$-matrices. Then, using $[\sigma_+, \sigma_-] = \sigma_z$, $[\sigma_z, \sigma_+] = 2\sigma_+$, $[\sigma_z, \sigma_-] = -2\sigma_-$, $[\sigma_i, I_{2\times 2}] = 0$ , we get:

$$\begin{aligned} \frac{1}{i\hbar}[H,\rho] &= \frac{-\omega_0}{2i}\left[\sigma_z,\, \rho_{ge} e^{i\omega t}\sigma_+ + \rho_{eg}\sigma_- e^{-i\omega t}\right] \\ &+ \frac{\omega_R}{2i}\left[\sigma_+ e^{i\omega t},\, \rho_{eg}\sigma_- e^{-i\omega t} + \frac{\rho_{gg}-\rho_{ee}}{2}\sigma_z\right] \\ &+ \frac{\omega_R}{2i}\left[\sigma_- e^{-i\omega t},\, \rho_{ge} e^{i\omega t}\sigma_+ + \frac{\rho_{gg}-\rho_{ee}}{2}\sigma_z\right] \\ &= \frac{-\omega_0}{2i}\left[2\rho_{ge} e^{i\omega t}\sigma_+ - 2\rho_{eg} e^{-i\omega t}\sigma_-\right] \\ &+ \frac{\omega_R}{2i}\left[\rho_{eg}\sigma_z - \frac{\rho_{gg}-\rho_{ee}}{2} 2\sigma_+ e^{i\omega t}\right] \\ &+ \frac{\omega_R}{2i}\left[\rho_{ge}\left(-\sigma_z\right) + \frac{\rho_{gg}-\rho_{ee}}{2} 2\sigma_- e^{-i\omega t}\right] \end{aligned}$$

or

$$\begin{aligned} \frac{1}{i\hbar}[H,\rho] &= \left(-\frac{\omega_0}{i}\rho_{ge} - \frac{\omega_R}{2i}\left(\rho_{gg}-\rho_{ee}\right)\right)\sigma_+ e^{i\omega t} \\ &+ \left(+\frac{\omega_0}{i}\rho_{eg} + \frac{\omega_R}{2i}\left(\rho_{gg}-\rho_{ee}\right)\right)\sigma_- e^{-i\omega t} \\ &+ \left(\omega_R \frac{\rho_{eg}-\rho_{ge}}{2i}\right)\sigma_z \end{aligned}$$

We can also reduce the spontaneous emission term to $\sigma$ matrices as:

$$-\Gamma \begin{pmatrix} -\rho_{ee} & \rho_{ge} e^{i\omega t}/2 \\ \rho_{eg} e^{-i\omega t}/2 & \rho_{ee} \end{pmatrix} = -\frac{\Gamma}{2}\rho_{ge} e^{i\omega t}\sigma_+ - \frac{\Gamma}{2}\rho_{eg} e^{-i\omega t}\sigma_- + \Gamma \rho_{ee}\sigma_z$$

Finally, under the given substitution

$$\dot{\rho} = \left(\dot{\rho}_{ge} + i\omega\rho_{ge}\right) e^{i\omega t}\sigma_+ + \left(\dot{\rho}_{eg} - i\omega\rho_{eg}\right)\sigma_- e^{-i\omega t} + \frac{\dot{\rho}_{gg} - \dot{\rho}_{ee}}{2}\sigma_z$$

Equating the $\sigma_i$ terms on both sides of the master equation and using $\partial_t\left(\rho_{gg} + \rho_{ee}\right) = 0$:

$$\begin{aligned}
\dot{\rho}_{ge} &= \left[i\left(\omega_0-\omega\right)-\frac{\Gamma}{2}\right]\rho_{ge}-\frac{\omega_R}{2i}\left(\rho_{gg}-\rho_{ee}\right)\\
\dot{\rho}_{eg} &= \left[i\left(\omega-\omega_0\right)\rho_{eg}-\frac{\Gamma}{2}\right]+\frac{\omega_R}{2i}\left(\rho_{gg}-\rho_{ee}\right)\\
\frac{\dot{\rho}_{gg}-\dot{\rho}_{ee}}{2}=\dot{\rho}_{gg}=-\dot{\rho}_{ee} &= \omega_R\frac{\rho_{eg}-\rho_{ge}}{2i}+\Gamma\rho_{ee}\sigma_z
\end{aligned}$$

This can be rewritten in terms of the detuning $\delta=\omega-\omega_0$ as:

$$\begin{aligned}
\dot{\rho}_{ge} &= \left(-i\delta-\frac{\Gamma}{2}\right)\rho_{ge}+i\frac{\omega_R}{2}\left(\rho_{gg}-\rho_{ee}\right)\\
\dot{\rho}_{eg} &= \left(+i\delta-\frac{\Gamma}{2}\right)\rho_{eg}-i\frac{\omega_R}{2}\left(\rho_{gg}-\rho_{ee}\right)\\
\dot{\rho}_{gg} &= +i\frac{\omega_R}{2}\left(\rho_{ge}-\rho_{eg}\right)+\Gamma\rho_{ee}\\
\dot{\rho}_{ee} &= -i\frac{\omega_R}{2}\left(\rho_{ge}-\rho_{eg}\right)-\Gamma\rho_{ee}
\end{aligned}$$

e) At large detuning, the excited state population will be small. Consequently, in the first two equations $\rho_{gg}-\rho_{ee}\approx 1$ and

$$\begin{aligned}
\dot{\rho}_{ge} &\approx \left(-i\delta-\frac{\Gamma}{2}\right)\rho_{ge}+i\frac{\omega_R}{2}\\
\dot{\rho}_{eg} &\approx \left(+i\delta-\frac{\Gamma}{2}\right)\rho_{eg}-i\frac{\omega_R}{2}
\end{aligned}$$

With the ansatz $\rho_{ge}=A+B\exp\left\{\left(-i\delta-\frac{\Gamma}{2}\right)t\right\}$ and requiring that at $t=0$ the atom starts unexcited ($\rho_{ge}\left(t=0\right)=0$) we get $B=-A$ and:

$$0=\left(-i\delta-\frac{\Gamma}{2}\right)A+i\frac{\omega_R}{2}$$

i.e.

$$\begin{aligned}
\rho_{ge} &\approx \frac{i\omega_R}{2\left(i\delta+\Gamma/2\right)}\left(1-\exp\left\{\left(-i\delta-\frac{\Gamma}{2}\right)t\right\}\right)\\
&= +\frac{i\omega_R}{2}\int_0^t\exp\left\{\left(-i\delta-\frac{\Gamma}{2}\right)\tau\right\}d\tau
\end{aligned}$$

and similarily

$$\begin{aligned}
\rho_{eg} &= \frac{-i\omega_R}{2\left(-i\delta+\Gamma/2\right)}\left(1-\exp\left\{\left(+i\delta-\frac{\Gamma}{2}\right)\right\}t\right)\\
&= -\frac{i\omega_R}{2}\int_0^t\exp\left\{\left(+i\delta-\frac{\Gamma}{2}\right)\tau\right\}d\tau
\end{aligned}$$

Substituting this in the equation for $\rho_{ee}$:

$$\begin{aligned}
\dot{\rho}_{ee} + \Gamma\rho_{ee} &= \left(-i\frac{\omega_R}{2}\right)\left(+\frac{i\omega_R}{2}\right)\int_0^t 2e^{-\Gamma\tau/2}\cos\left(\delta\tau\right)\, d\tau \\
&= \frac{\omega_R^2}{2}\int_0^t e^{-\Gamma\tau/2}\cos\left(\delta\tau\right)\, d\tau = f\left(t\right)
\end{aligned}$$

Taking the Laplace transform of both sides of the equation and using the properties of Laplace transforms together with $\rho_{ee}\left(0\right) = 0$ gives:

$$\begin{aligned}
\left(s+\Gamma\right)\mathcal{L}\left[\rho_{ee}\right]\left(s\right) &= \frac{\omega_R^2}{2}\frac{1}{s}\mathcal{L}\left[e^{-\Gamma t/2}\cos\left(\delta t\right)\right] \\
&= \frac{\omega_R^2}{2}\frac{1}{s}\mathrm{Re}\left\{\int_0^\infty e^{-st}e^{-\Gamma t/2+i\delta t}dt\right\} \\
\rho_{ee}\left(t\right) &= \frac{\omega_R^2}{2}\mathrm{Re}\left(\mathcal{L}^{-1}\left[g\left(s\right)\right]\left(t\right)\right) \\
g\left(s\right) &= \frac{1}{s}\frac{1}{s+\Gamma}\frac{1}{\Gamma/2+s-i\delta} = g\left(s\right)
\end{aligned}$$

To find the inverse Laplace transform of $g\left(s\right)$, use the Mellin transform with the vertical part of the contour to the right of $s = -\Gamma$. The contributions are from poles at $s = 0$, $s = -\Gamma$ and $s = -\Gamma/2 + i\delta$:

$$\begin{aligned}
g\left(t\right) &= \sum_{s_i}\mathrm{Res}\left(e^{st}g\left(s\right); s = s_i\right) \\
&= \frac{1}{\Gamma\left(\Gamma/2-i\delta\right)} + \frac{e^{-\Gamma t}}{-\Gamma}\frac{1}{-\Gamma/2-i\delta} + \frac{e^{-\Gamma t/2+i\delta t}}{-\Gamma/2+i\delta}\frac{1}{\Gamma/2+i\delta}
\end{aligned}$$

Taking the real part:

$$\begin{aligned}
\rho_{ee}\left(t\right) &= \frac{\omega_R^2}{2}\left(\frac{\Gamma/2}{\Gamma\left(\left(\Gamma/2\right)^2+\delta^2\right)} + \frac{e^{-\Gamma t}}{\Gamma}\frac{\Gamma/2}{\left(\Gamma/2\right)^2+\delta^2} - \frac{e^{-\Gamma t/2}\cos\left(\delta t\right)}{\delta^2+\left(\Gamma/2\right)^2}\right) \\
&= \frac{\omega_R^2}{4\left(\left(\Gamma/2\right)^2+\delta^2\right)}\left(1+e^{-\Gamma t}-e^{-\Gamma t/2}\cos\left(\delta t\right)\right) \\
\left(\delta \gg \Gamma\right) \quad &\approx \frac{\omega_R^2}{4\delta^2}\left(1+e^{-\Gamma t}-e^{-\Gamma t/2}\cos\left(\delta t\right)\right)
\end{aligned}$$

For $t \to \infty$, $\rho_{ee} \to \omega_R^2/\left(4\delta^2\right)$, which is in disagreement with our answer for part a). The reason for this is that spontaneousis a emission *does not* correspond to a friction force which damps the Rabi oscillations. Rather, it is a damping mechanism which pulls the Bloch vector towards the axis and towards the unexcited pole of the Bloch sphere.

f) In equilibrium, $\dot{\rho} = 0$ so

$$
\begin{aligned}
0 &= \left(-i\delta - \frac{\Gamma}{2}\right)\rho_{ge} + i\frac{\omega_R}{2}\left(\rho_{gg} - \rho_{ee}\right) \\
0 &= \left(+i\delta - \frac{\Gamma}{2}\right)\rho_{eg} - i\frac{\omega_R}{2}\left(\rho_{gg} - \rho_{ee}\right) \\
0 &= +i\frac{\omega_R}{2}\left(\rho_{ge} - \rho_{eg}\right) + \Gamma\rho_{ee}
\end{aligned}
$$

or

$$
\begin{aligned}
\Gamma\rho_{ee} &= -i\frac{\omega_R}{2}\left[\left(i\frac{\omega_R}{2}\right)/\left(i\delta + \Gamma/2\right) - \left(i\frac{\omega_R}{2}\right)/\left(i\delta - \Gamma/2\right)\right]\left(\rho_{gg} - \rho_{ee}\right) \\
&= -\left(i\frac{\omega_R}{2}\right)^2\left[-\frac{i\delta - \Gamma/2 - i\delta - \Gamma/2}{\delta^2 + \left(\Gamma/2\right)^2}\right]\left(\rho_{gg} - \rho_{ee}\right) \\
&= \frac{\omega_R^2}{2}\left[\frac{\Gamma/2}{\delta^2 + \left(\Gamma/2\right)^2}\right]\left(1 - 2\rho_{ee}\right)
\end{aligned}
$$

whence:

$$
\begin{aligned}
\rho_{ee} &= \frac{1}{2} \times \frac{\omega_R^2/2}{\delta^2 + \left(\Gamma/2\right)^2 + \omega_R^2/2} \\
R_{sc} &= \frac{\Gamma}{2} \times \frac{s}{\left(\delta/\left(\Gamma/2\right)\right)^2 + 1 + s}, \; s = \frac{\omega_R^2}{2\left(\Gamma/2\right)^2} = \frac{2\omega_R^2}{\Gamma^2}
\end{aligned}
$$

This can also be rewritten as

$$
R_{sc} = \frac{\Gamma}{2} \times \frac{s/\left(s+1\right)}{\left(\delta/\sqrt{1+s}\left(\Gamma/2\right)\right)^2 + 1}
$$

which corresponds to a Lorentzian of FWHM equal to $\sqrt{1+s} \times \Gamma$. This broadening of the atomic resonance due to Rabi oscillations is known as power broadening.

On resonance, we can also write the scattering rate as:

$$
R_{sc} = \frac{\Gamma}{2}\frac{s}{1+s} = \frac{\omega_R^2}{\Gamma} \times \frac{s}{1+s}, \; s = \frac{2\omega_R^2}{\Gamma^2}
$$

which is exactly the quantitative saturation behavior obtained by analogy with broadband excitation in Problem 2.

g) In this case the coherences decay with time constant $T_2$ while the populations decay with $T_1$. Tracing back the derivation of the evolution equations for $\rho_{ij}$'s , we can easily see that now:

$$
\begin{aligned}
\dot{\rho}_{ge} &= \left(-i\delta - \frac{1}{T_2}\right)\rho_{ge} + i\frac{\omega_R}{2}\left(\rho_{gg}-\rho_{ee}\right)\\
\dot{\rho}_{eg} &= \left(+i\delta - \frac{1}{T_2}\right)\rho_{eg} - i\frac{\omega_R}{2}\left(\rho_{gg}-\rho_{ee}\right)\\
\dot{\rho}_{gg} &= +i\frac{\omega_R}{2}\left(\rho_{ge}-\rho_{eg}\right) + \rho_{ee}/T_1\\
\dot{\rho}_{ee} &= -i\frac{\omega_R}{2}\left(\rho_{ge}-\rho_{eg}\right) - \rho_{ee}/T_1
\end{aligned}
$$

In equilibrium we now have:

$$
\begin{aligned}
\rho_{ee}/T_1 &= -i\frac{\omega_R}{2}\left(\rho_{ge}-\rho_{eg}\right)\\
&= -i\frac{\omega_R}{2}\left[\left(i\frac{\omega_R}{2}\right)/\left(i\delta + T_2^{-1}\right) - \left(i\frac{\omega_R}{2}\right)/\left(i\delta - T_2^{-1}\right)\right]\left(\rho_{gg}-\rho_{ee}\right)\\
&= -\left(i\frac{\omega_R}{2}\right)^2\left[-\frac{i\delta - 1/T_2 - i\delta - 1/T_2}{\delta^2 + \left(1/T_2\right)^2}\right]\left(\rho_{gg}-\rho_{ee}\right)\\
&= \frac{\omega_R^2}{2}\left[\frac{1/T_2}{\delta^2 + \left(1/T_2\right)^2}\right]\left(1-2\rho_{ee}\right)
\end{aligned}
$$

whence:

$$
\rho_{ee}\left(T_1^{-1} + \frac{\omega_R^2/T_2}{\delta^2 + T_2^{-2}}\right) = \frac{1}{2}\times\frac{\omega_R^2/T_2}{\delta^2 + \left(1/T_2\right)^2}
$$

or

$$
\rho_{ee} = \frac{1}{2}\times\frac{\omega_R^2 T_1 T_2}{\left(\delta T_2\right)^2 + 1 + \omega_R^2\, T_1 T_2}
$$

The $T_2$ decoherence time is seen to affect the linewidth while $T_1$ affects the saturation behavior of the NMR system with $s = \omega_R^2 T_1 T_2$.

We can also see that in the limit of large detuning,

$$
\rho_{ee} = \frac{1}{2}\times\frac{\omega_R^2}{\delta^2}\times\frac{T_1}{T_2}
$$

Setting $T_2 = 2T_1$ recovers the correct result for spontaneous emission while $T_1 = T_2$ reproduces the simple guess from part a).

Therefore, we see that the equilibrium excited state population will in general depend on the exact model of decoherence.