Physics 8.421 Spring 2008 Solution set for assignment #8
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1. **Line Shape Due to Fluctuating Field** We have a perturbing field that is not only oscillating at frequency ω, but also randomly flipping the sign of its amplitude at a mean rate Γ. In order to find the line shape, we start by finding the correlation fuction

$$G_{ba}(\tau) = \overline{\langle a|V(\tau)|b\rangle\langle b|V(0)|a\rangle}$$

Including natural decay, the dipole matrix element is

$$\langle a|V(\tau)|b\rangle = \frac{x}{2}e^{-\gamma|\tau|/2}e^{+i\omega\tau}$$
$$\langle b|V(0)|a\rangle = \frac{x}{2}$$

so that

$$G_{ba}(\tau) = \frac{1}{4}\overline{x(\tau)x(0)}e^{-\gamma|\tau|/2}e^{+i\omega\tau}$$

Now we need to find the system average $\overline{x(0)x(\tau)}$. Suppose that at $t = 0$ the field amplitude is $+A$. Then

$$\overline{x(0)x(\tau)} = A[Ap_+(\tau) - Ap_-(\tau)]$$

Since we know the mean rate of flipping, we can write rate equations for $p_+$and p_-:

$$\frac{dp_+}{d|\tau|} = \Gamma(p_- - p_+)$$
$$\frac{dp_-}{d|\tau|} = \Gamma(p_+ - p_-)$$
$$\frac{d}{d|\tau|}(p_+ - p_-) = -2\Gamma(p_+ - p_-)$$

With the initial condition, $p_+(0) = 1$ and $p_-(0) = 0$, we have $p_+(\tau) - p_-(\tau) = e^{-2\Gamma|\tau|}$, and

$$G_{ba}(\tau) = \frac{1}{4}A^2e^{-(2\Gamma+\frac{\gamma}{2})|\tau|}e^{+i\omega\tau}$$

The same result holds if the amplitude is $-A$ at $t = 0$, as can be seen by exchanging the p_+ and p_- and following through the same derivation.

We can now find the line shape.

$$\begin{aligned}
W_{ba} &= \int_{-\infty}^{+\infty} G_{ba}(\tau)e^{-i\omega_0\tau}d\tau \\
&= \frac{1}{4}A^2\int_{-\infty}^{+\infty} e^{-\beta|\tau|+i\delta\tau}d\tau \\
&= \frac{1}{4}A^2\int_{0}^{+\infty} (e^{-(\beta-i\delta)\tau}+e^{-(\beta+i\delta)\tau})d\tau \\
&= \frac{1}{4}A^2\frac{2\beta}{\beta^2+\delta^2} \\
&= \frac{1}{2}A^2\frac{2\Gamma+\frac{\gamma}{2}}{(2\Gamma+\frac{\gamma}{2})^2+(\omega-\omega_0)^2}
\end{aligned}$$

where $\beta = 2\Gamma + \frac{\gamma}{2}$ and $\delta = \omega - \omega_0$.

The lineshape is a Lorentzian with FWHM$= 4\Gamma + \gamma$.

2. **Intensity Distribution Due to Spontaneous Emission**

(a) The equality of the decay rates out of each (J, m) state gives

$$A = a = b + d = e + 2c$$

The equality of the rates into each (J', m') state gives

$$\frac{5}{3}A = a + b + c = e + 2d$$

Finally, the equality of rates for each photon polarization from an unpolarized mixture of excited states gives

$$a + c + d = e + 2b$$

In all three cases we have used the symmetry between m and $-m$ (which corresponds to spatial inversion symmetry) to account for the transitions that were not explicitly labelled in the problem.

Solving these equations gives

$$a = A \qquad d = \frac{1}{2}A$$
$$b = \frac{1}{2}A \qquad e = \frac{2}{3}A$$
$$c = \frac{1}{6}A$$

(b) The Wigner-Eckart Theorem is

$$\langle Jm_J\alpha|T_{Lm}|J'm_{J'}\alpha'\rangle = c(J'LJ; m_{J'}mm_J)\langle J\alpha \parallel T_L \parallel J'\alpha'\rangle$$

α and α' are the other quantum numbers not related to angular momentum. $\langle J\alpha \parallel T_L \parallel J'\alpha'\rangle$ is a quantity which is independent of $m_J, m_{J'}$ and m. The prefactor $c(J'LJ; m_{J'}mm_J)$, which is also often written as $\langle J'L; m_{J'}m|Jm_J\rangle$, is the Clebsch-Gordan coefficient for adding two angular momenta J' and L with z-components $m_{J'}$ and m, to get a resultant angular momentum J with z-component m_J.

In this case the operators T_{Lm} are

$$T_{1m}(\mathbf{r}) = \begin{cases} T_{11} = \frac{-1}{\sqrt{2}}(\boldsymbol{x} + i\boldsymbol{y}) = -\hat{\epsilon}_+ \cdot \boldsymbol{r} \\ T_{10} = \boldsymbol{z} = \hat{\epsilon}_\pi \cdot \boldsymbol{r} \\ T_{1-1} = \frac{1}{\sqrt{2}}(\boldsymbol{x} - i\boldsymbol{y}) = \hat{\epsilon}_- \cdot \boldsymbol{r} \end{cases}$$

Since the emission rates are proportional to the square of the corresponding matrix elements, we find that

$$\begin{aligned} a &\sim |\langle 1,1;1,1|2,2\rangle|^2 = 1 \\ b &\sim |\langle 1,1;1,0|2,1\rangle|^2 = 1/2 \\ c &\sim |\langle 1,1;1,-1|2,0\rangle|^2 = 1/6 \\ d &\sim |\langle 1,1;0,1|2,1\rangle|^2 = 1/2 \\ e &\sim |\langle 1,1;0,0|2,0\rangle|^2 = 2/3 \end{aligned}$$

up to an overall factor, which is just A.