Solutions 8.421 Midterm 2008

1.) HFS for positronium

a) Quantum #s

Low Field: n, $\ell = 0$; $S = S_e$, $I = S_p$; F, m_F (electron: S_e, positron: S_p)

<u>Note:</u> Since S and I are equivalent and have the same magnetic moment, S, I and L couple simultaneously. There is no "two-step" coupling of ℓ and S to J, and then J and I to F.

High Field: $n, \ell, s_e, s_p, m_\ell, m_{s,e}, m_{s,p}$

b) Low Field

1s $\quad S = F = 0, 1$ (singlet, triplet)

F = 0 state has $\mu = 0$.

F = 1: Spins parallel, magnetic moments antiparallel $\Rightarrow \mu = 0$

More Formally: all three m_F states $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ have $\langle \mu_z \rangle = 0$

High Field

$m_\ell = 0$

$m_S = m_F = m_{s,e} + m_{s,p}$

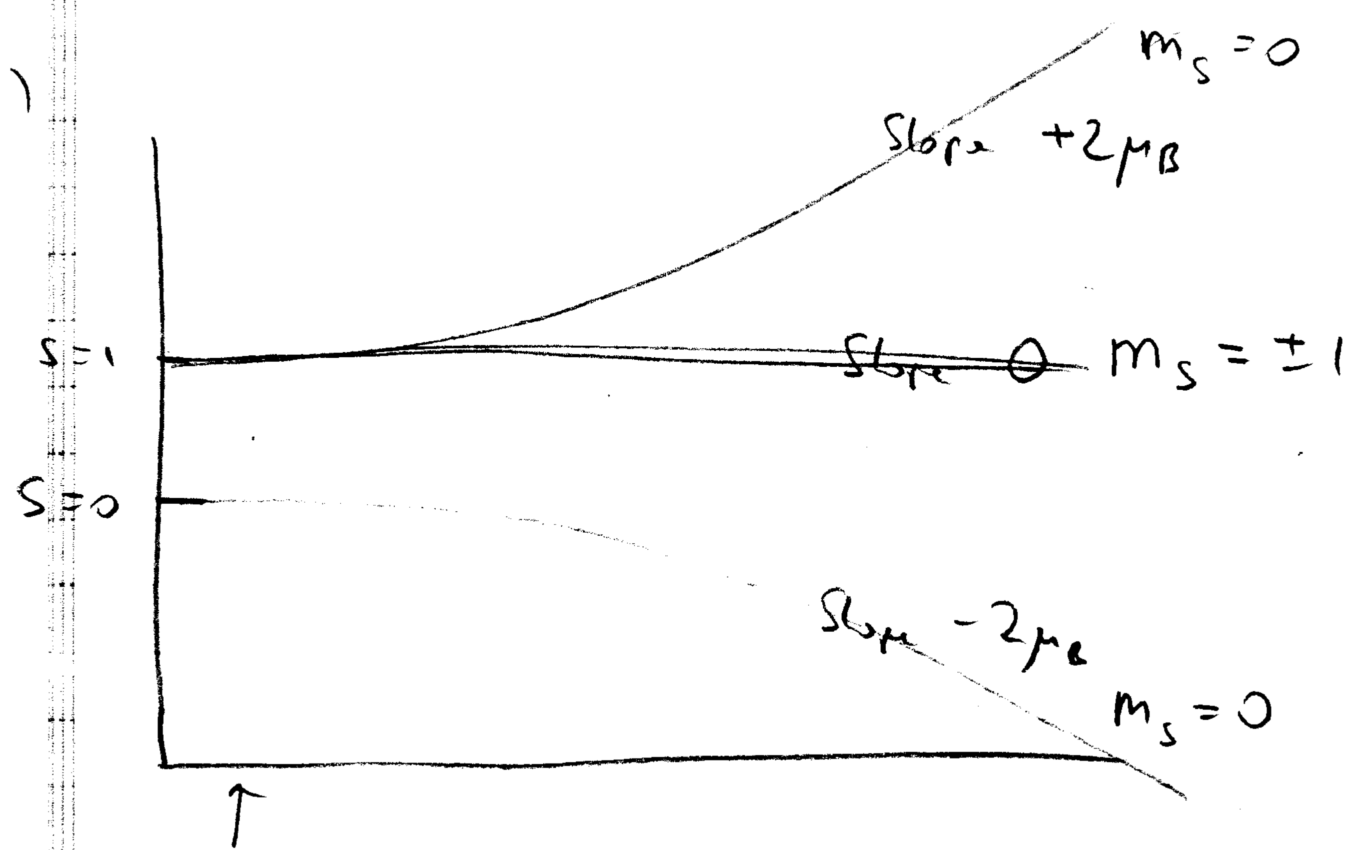
$m_S = \pm 1 \qquad |\uparrow\uparrow\rangle, \; |\downarrow\downarrow\rangle$

$\mu = 0$

$m_S = 0 \qquad |\uparrow\downarrow\rangle, \; |\downarrow\uparrow\rangle$

$\mu = \pm 2\mu_B$

c)



Lines connect states with the same $m_s = m_{s,e} + m_{s,p}$

d) $$E = \underbrace{2\mu_B\left(m_{s,e} - m_{e,p}\right)B}_{\text{Zeeman}} - a\, m_{s,e}\, m_{s,p}$$

e) The $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ states have $\mu = 0$ and therefore their energies don't depend on magnetic field.
$\Rightarrow$ They are degenerate at $B = 0$ and stay degenerate for all $B \neq 0$.

due to time reversal symmetry

2.) a) $n \quad l \quad m_l$

$n = 1, \ldots, \infty$
$l = 0, \ldots, (n-1)$
$m_l = -l, \ldots, +l$

$E = E(n)$ degeneracy in l, m_l

b) Finite size of e^-
<u>Raises</u> energy of S states
and removes degeneracy of S and $l \neq 0$
states. $n, l \neq 0$ states stay degenerate

c) $\vec{I} \cdot \vec{L}$ magnetic interaction
magnetic field of orbital motion couples
to $\mu_{nucleus}$

$\Rightarrow$ couples different m_l states
affects only $l \neq 0$
$\vec{L}$ and $\vec{I}$ coupled to $\vec{F}$

good quantum numbers n, l, F, m_F

d) $\vec{L} \cdot \vec{S}$ magnetic interaction
affects only $l \neq 0$
$\vec{L}$ and $\vec{S}$ couple to $\vec{J}$
n, l, J, m_J good quantum #s

relativistic: Darwin term + relativistic kinetic energy correction leads to degeneracy of Dirac equation $E = E(n, j)$ independent of l

3. Einstein rate eqs

a) Thermal equilibrium $\frac{N_e}{N_g} = e^{-\frac{\hbar\omega}{kT}}$

b) $\langle W(\omega)\rangle = \hbar\omega \, \langle n(\omega)\rangle \, \rho(\omega)$

$$\langle n(\omega)\rangle = \frac{1}{e^{\hbar\omega/kT} - 1} \qquad \text{for } d = 1, 2, 3$$

Density of states ρ : 2 polarizations

1D: $dn = 2\frac{dp\,dx}{h} = \frac{2\,dx}{2\pi\hbar}\frac{dE}{c} = \frac{d\omega\,dx}{\pi c}$

$$\rho_1 = \frac{dn}{dx\,d\omega} = \frac{1}{\pi c}$$

2D: $d^2n = \frac{2\,d^2p\,d^2x}{h^2} = \frac{2\,d^2x}{(2\pi)^2\hbar^2}\,2\pi\,p\,dp = \frac{d^2x}{\pi\hbar^2c^2}\,E\,dE = \frac{d^2x\,\omega}{\pi c^2}\,d\omega$

$$\rho_2 = \frac{d^2n}{d^2x\,d\omega} = \frac{\omega}{\pi c^2}$$

3D $d^3n = \frac{2\,d^3p\,d^3x}{h^3} = \frac{2\,d^3x}{(2\pi)^3\hbar^3}\,4\pi\,p^2dp = \frac{\omega^2\,d^3x}{\pi^2c^3}\,d\omega$

$$\rho_3 = \frac{d^3n}{d^3x\,d\omega} = \frac{\omega^2}{\pi^2c^3}$$

$$\langle W_1(\omega)\rangle = \frac{\hbar\omega}{\pi c}\langle n(\omega)\rangle = f_1\langle n(\omega)\rangle$$

$$\langle W_2(\omega)\rangle = \frac{\hbar\omega^2}{\pi c^2}\langle n(\omega)\rangle = f_2\langle n(\omega)\rangle$$

$$\langle W_3(\omega)\rangle = \frac{\hbar\omega^3}{\pi^2c^3}\langle n(\omega)\rangle = f_3\langle n(\omega)\rangle$$

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c) $N_g B_{ge} \langle W(\omega) \rangle = N_e (B_{eg} \langle W \rangle + A)$ detailed balance

$$e^{\frac{\hbar\omega}{kT}} = \frac{N_g}{N_e} = \frac{B_{eg}}{B_{ge}} + \frac{A}{B_{ge}\langle W \rangle} = \frac{B_{eg}}{B_{ge}} + \frac{A}{B_{ge} f \langle n(\omega) \rangle} =$$

$$= \frac{B_{eg}}{B_{ge}} + \frac{A}{B_{ge} f} (e^{\hbar\omega/kT} - 1) =$$

$$= \frac{B_{eg} - A/f}{B_{ge}} + \frac{A/f}{B_{ge}} e^{\hbar\omega/kT}$$

$\Rightarrow$ $A = f B_{eg}$ and $B_{eg} = B_{ge}$

1D $A = \frac{\hbar\omega}{\pi c} B_{eg}$

2D $A = \frac{\hbar\omega^2}{\pi c^2} B_{eg}$

3D $A = \frac{\hbar\omega^3}{\pi^2 c^3} B_{eg}$

4) Magic wavelength

a) Stark shift $\propto$ polarizability α.

$$\alpha_S = C\, d_{SP}^2 \left(\frac{1}{\omega_{SP} - \omega} + \frac{1}{\omega_{SP} + \omega} \right) = 2C \frac{d_{SP}^2\, \omega_{SP}}{\omega_{SP}^2 - \omega^2}$$

b) $$\alpha_P = C\, d_{SP}^2 \left(\frac{1}{-\omega_{SP} - \omega} + \frac{1}{-\omega_{SP} + \omega} \right) + C\, d_{PD}^2 \left(\frac{1}{\omega_{PD} - \omega} + \frac{1}{\omega_{PD} + \omega} \right)$$

$$= -2C \frac{d_{SP}^2\, \omega_{SP}}{\omega_{SP}^2 - \omega^2} + 2C \frac{d_{PD}^2\, \omega_{PD}}{\omega_{PD}^2 - \omega^2}$$

c) $\alpha_S \overset{!}{=} \alpha_P \Rightarrow \dfrac{2 d_{SP}^2\, \omega_{SP}}{\omega_{SP}^2 - \omega^2} = \dfrac{d_{PD}^2\, \omega_{PD}}{\omega_{PD}^2 - \omega^2}$

$$(\omega_{SP}^2 - \omega^2)\, d_{PD}^2\, \omega_{PD} = (\omega_{PD}^2 - \omega^2)\, 2 d_{SP}^2\, \omega_{SP}$$

$$(2 d_{SP}^2\, \omega_{SP} - d_{PD}^2\, \omega_{PD})\, \omega^2 = 2 d_{SP}^2\, \omega_{SP}\, \omega_{PD}^2 - d_{PD}^2\, \omega_{SP}^2\, \omega_{PD}$$

$$\omega^2 = \omega_{SP}\, \omega_{PD} \frac{2 d_{SP}^2\, \omega_{PD} - d_{PD}^2\, \omega_{SP}}{2 d_{SP}^2\, \omega_{SP} - d_{PD}^2\, \omega_{PD}}$$

$$= 2\omega_{PD}^2 \frac{8 d_{PD}^2\, \omega_{PD} - 2 d_{PD}^2\, \omega_{PD}}{16 d_{PD}^2\, \omega_{PD} - d_{PD}^2\, \omega_{PD}} = 2\omega_{PD}^2 \frac{6}{15} = \frac{4}{5} \omega_{PD}^2$$

$$\omega^2 = \frac{4}{5} \omega_{PD}^2$$

5. Transfer

i) z field static, $\frac{\pi}{2}$ pulse

ii) Larmor precession about static x field

iii) z field static, adiabatic change of ~~of~~ rotating field from −z to z

iv) field along −z, rotate slowly to +z

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