8.421 Spring 2004 (Prof. W. Ketterle) **4/7/2004**

Midterm Quiz

Note: All the problems are "quick" problems. Sometimes later parts of a problem can be answered, even if you miss the answer of an earlier part.

1. Energies of the hydrogen atom (4 points)

a. The Rydberg constant is Ryd=(1/2) $m_e\, e^4/\hbar^2$ where m_e is the mass of the electron. What are the energy levels (ignoring spin) of positronium where the proton (of the hydrogen atom) is replaced by a positron?

b. Use the expression for the Rydberg constant and derive the exact expression for the Bohr radius of the hydrogen atom (assuming the Bohr model).

2. Atoms in electric fields (14 points)

a. What is the "standard" interaction Hamiltonian H' for an atom in a static electric field?

b. In second order perturbation theory, the energy shift is $\sum_m \frac{|\langle m|H'|n\rangle|^2}{E_n - E_m}$.

What is the dc Stark shift for the ground state of an atom?
Explain the sign.

c. What are the induced dipole moment and the polarizability of the atom?

d. Give a criterion for the breakdown of perturbation theory.

e. How does the critical field for the breakdown depend on the principal quantum number n?

f. For an electromagnetic field of frequency ω close to resonance with an excited state: Modify the dc Stark effect formula for the ground state to describe the ac Stark shift. What modifications are necessary?
Assume $\vec{E}(t) = E_z \hat{e}_z \cos\omega t$

g. Try to write down (without derivation) the ac Stark shift for an electromagnetic field at arbitrary frequency!

3. Hyperfine structure at high magnetic fields (7 points)

a. What are the energy levels including hyperfine structure of an atom in very high magnetic field? The atom has no spin and is in a P state (L=1). The nuclear spin is I=1, and the hyperfine interaction Hamiltonian is $ha\vec{I}\vec{L}$. In your answer, you may use the Bohr magnetron μ_B, nuclear magnetron μ_N and g_I for the nuclear g factor.

b. What is the Hamiltonian which couples the high-field eigenstates and becomes important at low field?

c. Explain in words (two sentences …), how you would calculate the hyperfine constant a.

4. Spontaneous emission (5 points)

a. An atom has an excited P level which decays to an S state. Are the lifetimes for spontaneous decay of the three P states the same, or are the different? Say why.

b. A (weak) external magnetic field provides the quantization axes along $\hat{z}$. Assume now that the atom is embedded in a photonic medium, which forces the x component of the electric field to vanish, but does not change the density of modes or other properties. How do the lifetimes of the P states change?

5. The $^3He^+$ atom (10 points)

a. An electron is in the ground state of tritium, for which the nucleus consists of a proton and two neutrons. A nuclear reaction instantaneously changes the nucleus to $^3He^{++}$, that is, two protons and one neutron. What is the expression (just the formula, no numbers) for the probability that the electron remains in the ground state of $^3He^+$?

b. What states of $^3He^+$ should have the highest probability of occupation? For this, write down an exact expression for the expectation value for the radial coordinate $\left\langle \frac{1}{r} \right\rangle$ for the initial state and the states of $^3He^+$.

c. When the final state is the 2s state: How can the atom decay by one photon emission?
Consider (with one brief sentence) the possibilities of E1, E2 and M1 radiation? Will it make a difference whether the electron has spin or not?

Midterm Quiz 8.421 Spring 2004 1

Solutions

1 a) $(m_e)_{eff} \rightarrow \frac{1}{2}(m_e)_{eff}$ reduced mass

(2)

$$\Rightarrow Ryd\Big|_{Positron} = \frac{1}{2} Ryd$$

$$E_n = -\frac{1}{2} Ryd \frac{1}{n^2} \quad \text{For positronium}$$

4 points b) $E_1 = -Ryd = -\frac{1}{2}\frac{m_e e^4}{\hbar^2}$

(2)

$$E_1 = \frac{1}{2}\frac{-e^2}{a_0} \quad \text{Virial theorem}$$

$$\stackrel{!}{=} -\frac{1}{2}\frac{m_e e^4}{\hbar^2}$$

$$\Rightarrow \boxed{a_0 = \frac{\hbar^2}{m_e e^2}}$$

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2 a) $H' = -e\vec{r}\cdot\vec{E}$

(1) ↳ Operator

b) $\Delta E_0 = -\sum \frac{|\langle m|H'|0\rangle|^2}{E_m}$

14 points (1)

(1) negative sign : ground state is repelled

c) induced dipole moment: $\Delta E_0 = \frac{1}{2}\alpha E^2$

(2) $= \frac{1}{2} D E$

(2) d) Matrix element $\langle m|e\vec{r}\cdot\vec{E}|n\rangle \gtrsim E_m - E_n$

(3) e) $E_m - E_n \propto \frac{1}{n^3}$ $\quad \langle r\rangle \propto n^2 \Rightarrow E_{crit} \propto n^{-5}$

With $H' = -e z E_{z0}$

f)

$$\Delta E_0 = -\frac{1}{2} \frac{|\langle m | H' | 0 \rangle|^2}{E_m - \hbar\omega}$$

(2) only one term out of two (rot. wave approximation)

frequency $\hbar\omega$

(2) g) $$\Delta E = -\frac{1}{2} \sum_m |\langle m | H' | 0 \rangle|^2 \left(\frac{1}{E_m - \hbar\omega} + \frac{1}{E_m + \hbar\omega} \right)$$

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3 a) $$E = E_p + \mu_B \cdot B\, m_\ell - g_I \mu_N B\, m_I + ha\, m_I m_L$$

(3)

7 points

(2) b). $ha (I_x L_x + I_y L_y)$ couples m_I, m_L states

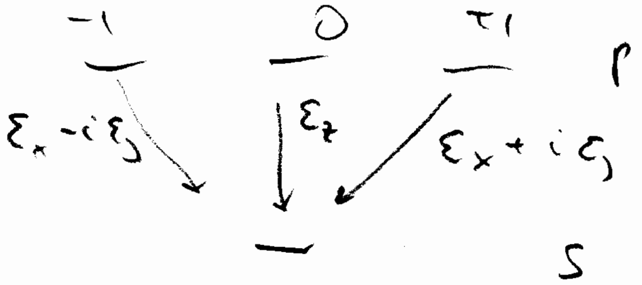
c). Biot-Savart Law for orbiting electron.
(2) Creates magnetic field at the origin $\vec{B}_L$
Hyperfine energy is $\vec{\mu}_{nuclear} \cdot \vec{B}_L$

(3

4 a) P Level $m = 0$, $m = \pm 1$

(2) same lifetime; rotation symmetry

b)



5 points (3)

No $\mathcal{E}_x$ field: reduces matrix elements for $\Delta m = \pm 1$ transition by $\sqrt{2}$ and increases the lifetime by a factor of 2.

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5 (2) a) $|\langle 1s|_{Hyd} \,|\, 1s|_{He}\rangle|^2$. Overlap matrix element

(2) b) $\langle \frac{1}{r} \rangle = \frac{Z}{n^2 a_0}$

10 points (2) Hydrogen $Z=1$ compared to helium $Z=2$; 1s state has a piece between the 1s and 2s states of helium. Therefore one would expect those states to be preferably populated.

(4) c)

E1 always forbidden by parity

E2 triangle rule forbids $J = 0, \frac{1}{2} \to 0, \frac{1}{2}$

M1 forbidden for $J = 0$ no electron spin

but very weak ⟶ Allowed for $J = \frac{1}{2}$ with electron spin