8.421 Spring 2006 (Prof. W. Ketterle) **4/5/2006**

Midterm Quiz

Note: All the problems are "quick" problems. Sometimes later parts of a problem can be answered, even if you miss the answer of an earlier part.

1. Hydrogenic ions with nuclear charge Z (11 points)

a. Write down the non-relativistic Hamiltonian for a hydrogenic ion with nuclear charge Z. By rescaling length and energy, you should be able to transform to the Hamiltonian for hydrogen. What are the scaling relations?

Find the scaling with Z of

b. The expectation values of r, 1/r and $1/r^3$ where r is the distance of the electron from the nucleus,

c. The expectation value of the potential energy V,

d. The total energy E,

e. The velocity of the electron,

f. The probability to find the electron at the origin, $|\psi(r=0)|^2$,

g. The fine structure splitting.

h. The hyperfine structure energy splitting due the magnetic dipole of the nucleus (which should be regarded as independent of Z).

2. Rydberg atoms in electric fields (6 points)

a. For a Rydberg atom with principal quantum number n, how does the size scale with n?

b. How does the energy scale with n?

c. How does the density of closely lying states scale with n?

d. What is the perturbative expression for the polarizability?

e. How does the polarizability scale with n?

3. Paschen Back effect of the fine structure at high magnetic fields (8 points)

An atom has orbital angular momentum L, spin angular momentum S, and no nuclear spin. The fine structure interaction Hamiltonian is $A\vec{S}\vec{L}$

a. What are the relevant quantum numbers at low and very high magnetic fields?

b. What are the energy levels of an atom in very high magnetic field? In your answer, you may use the Bohr magnetron μ_B.

c. What are the energy levels of an atom in zero magnetic field?

4. Spontaneous emission in two dimensions (12 points)

In class, we derived the expression for the absorption rate in a two-level atom for an electromagnetic field with frequency ω_0 and polarization ε. $n(\omega)\, d\omega$ is the number of photons in the interval $d\omega$, and V the volume of the electromagnetic field:

$$\Gamma_{ba} = \frac{4\pi^2}{\hbar V}\left|\hat{\varepsilon} \bullet \vec{D}_{ba}\right|^2 \omega_0 n(\omega_0)$$

Find the spontaneous lifetime for an atom that interacts with a two-dimensional electromagnetic field. Assume that the atom is in a parallel plate capacitor with plate separation d<<λ, the resonant wavelength.

a. Discuss the relevant modes in the parallel plate capacitor, and calculate the mode density.

b. Calculate the rate of spontaneous decay without evaluating or discussing . $\vec{D}_{ba}$. Express the two-dimensional lifetime in terms of the three-dimensional lifetime. Try to explain why the 2D lifetime is larger or smaller than the 3D lifetime.

c. Discuss now both s to p and p to s transitions by considering $\vec{D}_{ba}$.

Midterm quiz 8.421 2006

Solutions

1. a) $H = \frac{\hbar^2}{2m} \nabla^2 - \frac{Ze^2}{r}$

$\boxed{3}$ $r \rightarrow \rho = r/Z$

$\nabla^2 \rightarrow Z^2 \nabla_\rho^2$

$H = Z^2 \left(-\frac{\hbar^2}{2m} \nabla_\rho^2 - \frac{e^2}{\rho} \right)$

r scales with Z^{-1}

energy scales with Z^2

b) $\boxed{1}$ $\langle r^n \rangle \propto Z^{-n}$

c) $\boxed{1}$ $\langle V \rangle \propto Z^2$

d) $\boxed{1}$ $T = -\frac{1}{2} \langle V \rangle$ Kinetic energy

$\Rightarrow \langle E \rangle \propto Z^2$

e) $\boxed{1}$ $v \propto \sqrt{T} \propto Z$

f) $\boxed{1}$ $|\psi(r=0)|^2$ has dimension of length^{-3}

$\Rightarrow |\psi(r=0)|^2 \propto Z^{+3}$

h) $\boxed{1}$ $\Delta E_{hf} \propto |\psi(0)|^2 \propto Z^3$

j) $\boxed{2}$ magnetic field B due to relative motion

$B = |\vec{v} \times \vec{E}|/c$ $\qquad \mathcal{E} = \frac{Ze}{r^2} \propto Z^4$

$B \propto Z^4$

2.) a) $\boxed{1}$ $\langle r \rangle \propto n^2$

Follows from Bohr Formula

$E \propto \frac{1}{r} \propto \frac{1}{n^2}$

b) $\boxed{1}$ $E_n \propto \frac{1}{n^2}$

c) $\boxed{1}$ energy density $\frac{dE}{dn} \propto \frac{1}{n^3}$

d) $\boxed{1}$ $\Delta E_k = \sum_i \frac{d_{ik}^2 \mathcal{E}^2}{E_k - E_i} = \frac{1}{2} \propto \mathcal{E}^2$

e) $\boxed{2}$ $d_{ik} \propto n^2$

$E_k - E_i \propto n^{-3}$

$\propto \quad \propto \underline{n^7}$

3.) a) $\boxed{2}$ low field L, S, J, M_J

high field L, S, M_L, M_S

b) $\boxed{3}$ $E = E_0 + \mu_B \cdot B\,(M_L + 2M_S) + A\,M_S M_L$

c) $\boxed{3}$ $E = E_0 + A\,\vec{S}\vec{L}$

$(S+L)^2 = J^2 \Rightarrow \vec{S}\vec{L} = \frac{1}{2}(J^2 - S^2 - L^2)$

$E = E_0 + \frac{A}{2}\left(J(J+1) - S(S+1) - L(L+1)\right)$

4.) a) $E_{trans} = 0$ at boundaries $\Rightarrow E_{x,y} = 0$

$E_z = f(x,y) \Rightarrow$ 2D problem

(any mode along z would require $\omega = k \cdot c \geq \frac{\pi}{d} c \gg \omega_{res}$)

\# of modes: $dN = dk_x \, dk_y \cdot \frac{A}{(2\pi)^2} = \frac{A}{(2\pi)^2} \frac{\omega \, d\omega}{c^2} \, d\Theta$

$\boxed{4}$

A: area

$$\frac{dN}{d\omega} = \frac{A}{(2\pi)^2} \frac{\omega}{c^2} \cdot 2\pi$$

b) $\Gamma_{ba} = \frac{4\pi^2}{\hbar V} \omega_0 \left| \varepsilon D_{ba} \right|^2 \frac{dN}{d\omega}$

$\boxed{4}$ $= \frac{\omega_0^2 \cdot 2\pi}{\hbar c^2 d} \left| \hat{z} \cdot \vec{D}_{ba} \right|^2$

$A/V = 1/d$

Two-level atom:

Case 1: $\vec{D}_{ba} \parallel$ surface

$\Gamma_{ba} = 0$

Case 2: $\vec{D}_{ba} \perp$ surface

$\left| \hat{z} \cdot \vec{D}_{ba} \right| = \left| \vec{D}_{ba} \right|$

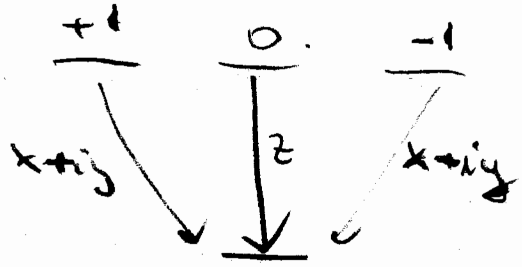
For Case 2: $\Gamma_{ba} = \frac{\omega_0^2 \cdot 2\pi}{\hbar c^2 d} \left| \vec{D}_{ba} \right|^2$

in 3D $\Gamma_{ba} = \frac{4}{3} \frac{\omega_0^3}{\hbar c^3} \left| \vec{D}_{ba} \right|^2$

$$\Rightarrow \boxed{\Gamma_{ba}^{2D} = \Gamma_{ba}^{3D} \times \frac{3}{4} \frac{\lambda}{d}} \gg \Gamma_{ba}^{3D} \text{ for } \lambda \gg d$$

c) $\boxed{4}$

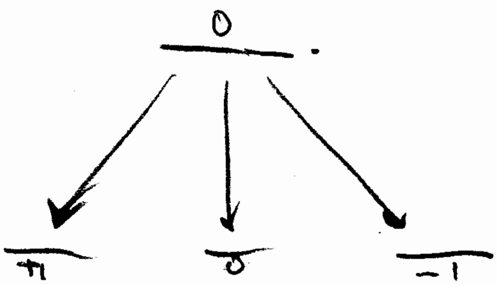
$p \rightarrow s$



$$\Gamma_{m=+1} = \Gamma_{m=-1} = 0$$

$$\Gamma^{2D}_{m=0} = \Gamma^{3D}_{m=0} \times \frac{3}{4}\frac{\lambda}{d} \gg \Gamma^{3D}_{ba} \quad \text{for } \lambda \gg d$$

$s \rightarrow p$



in 3D: 3 possible transitions

in 2D only one ($\Delta m = 0$)

$$\Rightarrow \Gamma^{2D}_{s} = \underbrace{\Gamma^{3D}_{s} \times \frac{3}{4}\frac{\lambda}{d}}_{\text{as above}} \times \frac{1}{3}$$

Belongs to b:

The 2D decay rate is either zero (no modes to carry away the photon) or faster (because of the smaller volume per mode $\Rightarrow$ 1 photon per mode in the capacitor has a higher electric field in 2D than in 3D).