

Assignment #5

Due: Friday, March 24, 2006

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Office hours from 1 to 3 on Thursday, March 23, in room 26-201.

1. DC Stark effect for an alkali atom

The oscillator strength f_{kj} between atomic levels k and j with frequency difference ω_{kj} is defined as

$$f_{kj} = \frac{2m}{3\hbar e^2} \omega_{kj} \frac{1}{2J_j + 1} \sum_{m_k, m_j} |\langle k, m_k | \mathbf{D} | j, m_j \rangle|^2, \quad (1)$$

where m and e are, respectively, the mass and charge of the electron. For alkali atoms, essentially all of the oscillator strength out of the ground state comes from the $nS \rightarrow nP$ transition (i.e. $f_{nP, nS} = 1$). Thus, all you need to know about an alkali atom is its transition frequency ω_0 . Using just these facts, find the critical electric field, ϵ_c , beyond which the Stark effect is no longer quadratic (where $\langle nP | H' | nS \rangle = \hbar\omega_0$). Express the result in V/cm where ω_0 is expressed in atomic units. What is ϵ_c for sodium (resonant wavelength $\lambda = 589$ nm)?

2. Oscillator strengths and experimental measurements

The experimental lifetime of the Na $3P$ state is 16.6 ns and for the H $2P$ state it is 1.60 ns. From these, calculate $f_{3S, 3P}$ and $f_{3P, 3S}$ for Na, and $f_{1S, 2P}$ and $f_{2P, 1S}$ for H. **Note:** (1) These are oscillator strengths between levels, not between specific m states; (2) Oscillator strength is positive (negative) for absorption (emission) – pay attention to the signs.

3. Optical traps and scattering

The AC Stark effect has been used to trap neutral atoms at the focus of one or more laser beams. In the following exercise, you will explore the proper laser power and wavelength needed to trap an ultracold atomic gas.

Let us again consider an alkali atom with resonance frequency ω_0 on the principal $nS \rightarrow nP$ transition. A sample of atoms in the ground state (nS) are exposed to monochromatic radiation of intensity I and frequency $\omega_L < \omega_0$ (red-detuned from resonance).

a) Calculate the AC Stark shift using both

- i.* the perturbative result for the dynamic polarizability $\alpha(\omega)$, and
- ii.* the nonperturbative approach in the rotating wave approximation (as treated in section 1.7 of the course notes and problem 2 from the first homework), but then approximated for small field strengths.

Calculate the ratio of results (*i*) / (*ii*) as a function of ω_L .

b) What is the likelihood of finding the atom in the excited state using both pictures (*i*) and (*ii*)? Neglect rapidly oscillating terms.

- c) What is the photon scattering rate using (i) and (ii) as a function of ω_L ? [Check: does your answer give the blue-sky formula (scattering power proportional to ω_L^4)?]

Finally, let us apply our formulae to the optical trapping of sodium. Atoms are to be trapped at the focus of a single laser beam. At the focal plane, the irradiance distribution has the form

$$I(r) = \frac{2P}{\pi w^2} \exp(-2r^2/w^2) \quad (2)$$

where $w = 6 \mu\text{m}$ is called the beam waist radius, r is the distance away from the center of the beam, and P is the laser power. Such a beam forms a trap for which the trap depth is given by the AC Stark shift at the maximum intensity $I(0)$.

- d) We wish to trap a gas of sodium atoms with a temperature of about $1 \mu\text{K}$. Thus, let us plan for a trap depth of $k_B \times 10 \mu\text{K}$. Calculate the required power and the scattering rate for the following laser options:
1. **Yellow Option:** a detuning of 1.7 GHz (light at such a detuning is used for cooling sodium in the first place).
 2. **Red Option:** the light of a He-Ne laser (633 nm). Incidentally, a run-of-the-mill laser pointer can put out 1 mW of power.
 3. **Infrared Option:** the light of a diode laser at 985 nm.

You should use approach (i) which includes the counter-rotating term. For the Infrared Option, how much of a difference does this term make as a percentage of the total power needed and the scattering rate?

For reference:

Steven Chu *et al.* “Experimental observation of optically trapped atoms,” PRL **57**, 314 (1986). First optical dipole trap for laser-cooled sodium atoms.

D.M. Stamper-Kurn *et al.* “Optical confinement of a Bose-Einstein condensate,” PRL **80**, 2027 (1998). First optical dipole trap for Bose condensed sodium atoms.

4. Atom-Field Interaction Hamiltonian

In classical electromagnetic theory, a gauge transformation on the scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ is defined by the relations

$$\phi'(\mathbf{r}, t) = \phi(\mathbf{r}, t) - \frac{\partial}{\partial t} f(\mathbf{r}, t) \quad (3)$$

$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \nabla f(\mathbf{r}, t) \quad (4)$$

where $f(\mathbf{r}, t)$ is an arbitrary differentiable function.

A gauge transformation is applied in quantum mechanics as a unitary transformation $O(\mathbf{r}, t)$ (for which $OO^\dagger = 1$) which transforms states as

$$\psi'(\mathbf{r}, t) = O(\mathbf{r}, t)\psi(\mathbf{r}, t). \quad (5)$$

Such a transformation transforms the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H\psi(\mathbf{r}, t) \quad (6)$$

into

$$i\hbar \frac{\partial}{\partial t} \psi'(\mathbf{r}, t) = H'\psi'(\mathbf{r}, t). \quad (7)$$

a) How is H' related to H and $O(\mathbf{r}, t)$?

Now let us consider the interaction Hamiltonian (setting $c = 1$)

$$H^{int}(\mathbf{r}, t) = \frac{1}{2m}[\mathbf{p} + q\mathbf{A}(\mathbf{r}, t)]^2 - q\phi(\mathbf{r}, t). \quad (8)$$

We introduce a gauge transformation by using the unitary operator

$$O(\mathbf{r}, t) = e^{-iqf(\mathbf{r}, t)/\hbar}. \quad (9)$$

b) How is H^{int} transformed by this gauge transformation? You should be able to identify the gauge transformation explicitly as defined in Eqs. (2) and (3).

Now, let us consider the case of no scalar potential ($\phi = 0$). The field is then described completely by the vector potential $\mathbf{A}(\mathbf{r}, t)$. We choose a gauge transformation of the form

$$f(\mathbf{r}, t) = -\mathbf{A}(t) \cdot \mathbf{r}. \quad (10)$$

In the spirit of the electric dipole approximation, let us neglect the spatial dependence of $\mathbf{A}(t)$ (meaning the wavelength of radiation is much larger than the size of our atom).

c) Find the transformed $H^{int'}$. Identify the electric field.

You may recall that in class, we adopted a different approach, where we expanded the interaction Hamiltonian as

$$H^{int} = -\frac{q}{m}\mathbf{p} \cdot \mathbf{A} + \frac{q^2}{2m}|\mathbf{A}^2| \quad (11)$$

$$= H^{(1)} + H^{(2)} \quad (12)$$

We neglected the term $H^{(2)}$ because as a two-photon process, it contributes only at high intensities and for certain photon-scattering processes. As you have just shown, this step can be avoided using a gauge transformation. Also, notice that higher order terms can be obtained by considering the spatial dependence of the vector potential.