Homework Assignment #7

Physics 8.421, Spring 2006, Prof. W. Ketterle

Due Friday, April 14

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1. Optical Bloch Equations with Spontaneous Emission

Consider a two level system driven at large detuning $|\delta| \gg \Gamma, \omega_R$. The system is in the ground state , $|a\rangle$, at time $t = 0$.

a) For $\Gamma = 0$, what is the excited state fraction, $\rho_{bb}(t)$, given by the solution for undamped Rabi oscillations?

What do you expect will happen if a weak damping term is added to account for spontaneous emission? Guess the result for ρ_{bb} in the limit $t \to \infty$.

b) The evolution of the system, including damping effects, can be determined from the optical Bloch equations. With a rate of spontaneous emission, Γ, the equation of motion for the density matrix is:

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] - \Gamma \begin{pmatrix} -\rho_{bb} & \rho_{ab}/2 \\ \rho_{ba}/2 & \rho_{bb} \end{pmatrix}$$

where $H = \frac{\hbar}{2} \begin{pmatrix} -\omega_0 & \omega_R e^{i\omega t} \\ \omega_R e^{-i\omega t} & \omega_0 \end{pmatrix}$ and $\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$ with $\rho_{ab} = \rho_{ba}^*$ and normalization condition $\rho_{aa} + \rho_{bb} = 1$.

First, solve for the time-independent form of the optical Bloch equations by making the substitution $\hat{\rho}_{ab} = \rho_{ab} e^{-i\omega t}$ and $\hat{\rho}_{ba} = \rho_{ba} e^{i\omega t}$ to get:

$$\begin{aligned}
\dot{\rho}_{aa} &= i\tfrac{\omega_R}{2}(\hat{\rho}_{ab} - \hat{\rho}_{ba}) + \Gamma\rho_{bb} \\
\dot{\rho}_{bb} &= -i\tfrac{\omega_R}{2}(\hat{\rho}_{ab} - \hat{\rho}_{ba}) - \Gamma\rho_{bb} \\
\dot{\hat{\rho}}_{ab} &= (-i\delta - \tfrac{\Gamma}{2})\hat{\rho}_{ab} + i\tfrac{\omega_R}{2}(\rho_{aa} - \rho_{bb}) \\
\dot{\hat{\rho}}_{ba} &= (i\delta - \tfrac{\Gamma}{2})\hat{\rho}_{ba} - i\tfrac{\omega_R}{2}(\rho_{aa} - \rho_{bb})
\end{aligned}$$

Next, show that in the limit $|\delta| \gg \Gamma, \omega_R$ and with initial conditions $\rho(t=0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the optical Bloch equations give:

$$\rho_{bb} = \frac{\omega_R^2}{4\delta^2}\left(1 + e^{-\Gamma t} - 2e^{-\Gamma t/2}\cos\delta t\right)$$

What does this reduce to in the limit $t \to \infty$?

c) Show that the steady state solution for arbitrary δ, Γ, and ω_R is:

$$\rho_{bb} = \frac{\omega_R^2}{4}\frac{1}{\delta^2 + \Gamma^2/4 + \omega_R^2/2}$$

What does this reduce to in the limit of large detuning?

d) Now calculate ρ_{bb} using time-independent perturbation theory. For this, assume that the two states, $|a, n \text{ photons}\rangle$ and $|b, n-1 \text{ photons}\rangle$, are coupled by $H_{int} = -\vec{d}\cdot\vec{E}$ with

$$\vec{E} = i\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{\epsilon}(a - a^\dagger).$$

(This is the Schrödinger representation for $\vec{E}$ in the dipole approximation.) Identify the Rabi frequency as

$$\hbar\omega_R = 2\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{\epsilon} \cdot \vec{d}_{ab}\sqrt{n}.$$

(Note that for $n = 1$, ω_R is the vacuum Rabi frequency.) Compare your result with part c).

e) Find the steady state solution for ρ_{bb} for the following Bloch equation:

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + \begin{pmatrix} \Gamma_1\rho_{bb} & -\Gamma_2\rho_{ab} \\ -\Gamma_2\rho_{ba} & -\Gamma_1\rho_{bb} \end{pmatrix}$$

2. Cross Section for Absorption

The cross section for absorption of laser radiation is defined as the area per atom which scatters (absorbs) an incident beam of photons. Equivalently,

$$\sigma(\omega) = \frac{\text{absorption rate}}{\text{incident photon flux}}.$$

The monochromatic unsaturated absorption rate for a two level system can be shown to be

$$\Gamma_{ab} = \frac{\omega_R^2}{A_{ba}}\left[1 + \left(\frac{2(\omega - \omega_{ba})}{A_{ba}}\right)^2\right]^{-1},$$

where A_{ba} is the spontaneous emission rate from b to a.

Find the cross section for absorption on resonance, $\sigma(\omega = \omega_{ba})$. Assume the lower state a has $J_a = 0$, i.e. a multiplicity of 1. Calculate the cross section for a transition with wavelength $\lambda = 500$ nm.