Midterm quiz 8.421 2006
Solutions

1. a) $H = \frac{\hbar^2}{2m} \nabla^2 - \frac{Z e^2}{r}$

$\boxed{3}$ $r \to \rho = r/Z$

$\nabla^2 \to Z^2 \nabla_\rho^2$

$H = Z^2 \left(-\frac{\hbar^2}{2m} \nabla_\rho^2 - \frac{e^2}{\rho} \right)$

r scales with Z^{-1}

energy scales with Z^2

b) $\boxed{1}$ $\langle r^n \rangle \propto Z^{-n}$

c) $\boxed{1}$ $\langle V \rangle \propto Z^2$

d) $\boxed{1}$ $T = -\frac{1}{2} \langle V \rangle$ Kinetic energy

$\Rightarrow \langle E \rangle \propto Z^2$

e) $\boxed{1}$ $v \propto \sqrt{T} \propto Z$

f) $\boxed{1}$ $|\psi(r=0)|^2$ has dimension of length^{-3}

$\Rightarrow |\psi(r=0)|^2 \propto Z^{+3}$

h) $\boxed{1}$ $\Delta E_{hf} \propto |\psi(0)|^2 \propto Z^3$

j) $\boxed{2}$ magnetic field B due to relative motion

$B = |\vec{v} \times \vec{E}|/c$ $\quad \mathcal{E} = \frac{Ze}{r^2} \propto Z^4$

$B \propto Z^4$

2.) a) $\boxed{1}$ $\langle r \rangle \propto n^2$

Follows from Bohr Formula

$$E \propto \frac{1}{r} \propto \frac{1}{n^2}$$

b) $\boxed{1}$ $E_n \propto \frac{1}{n^2}$

c) $\boxed{1}$ energy density $\frac{dE}{dn} \propto \frac{1}{n^3}$

d) $\boxed{1}$ $\Delta E_k = \sum_i \frac{d_{ik}^2 \mathcal{E}^2}{E_k - E_i} = \frac{1}{2} \propto \mathcal{E}^2$

e) $\boxed{2}$ $d_{ik} \propto n^2$

$E_k - E_i \propto n^{-3}$

$\propto \quad \propto \underline{n^7}$

3.) a) $\boxed{2}$ low field L, S, J, M_J

high field L, S, M_L, M_S

b) $\boxed{3}$ $E = E_0 + \mu_B \cdot B (M_L + 2 M_S) + A M_S M_L$

c) $\boxed{3}$ $E = E_0 + A \vec{S} \vec{L}$

$$(S+L)^2 = J^2 \Rightarrow \vec{S}\vec{L} = \frac{1}{2}(J^2 - S^2 - L^2)$$

$$E = E_0 + \frac{A}{2}\left(J(J+1) - S(S+1) - L(L+1) \right)$$

4.) a) $E_{trans} = 0$ at boundaries $\Rightarrow E_{x,y} = 0$

$E_z = f(x,y) \Rightarrow$ 2D problem

(any mode along z would require $\omega = k \cdot c \geq \frac{\pi}{d} c \gg \omega_{res}$)

$\boxed{4}$ # of modes: $dN = dk_x \, dk_y \cdot \frac{A}{(2\pi)^2} = \frac{A}{(2\pi)^2} \frac{\omega \, d\omega}{c^2} d\Theta$

A: area

$$\frac{dN}{d\omega} = \frac{A}{(2\pi)^2} \frac{\omega}{c^2} \cdot 2\pi$$

b) $\Gamma_{ba} = \frac{4\pi^2}{\hbar V} \omega_0 \left| \varepsilon D_{ba} \right|^2 \frac{dN}{d\omega}$

$\boxed{4}$ $= \frac{\omega_0^2 \cdot 2\pi}{\hbar c^2 d} \left| \hat{z} \cdot \vec{D}_{ba} \right|^2$

$A/V = 1/d$

Two-level atom:

Case 1: $\vec{D}_{ba} \parallel$ surface

$\Gamma_{ba} = 0$

Case 2: $\vec{D}_{ba} \perp$ surface

$\left| \hat{z} \cdot \vec{D}_{ba} \right| = \left| \vec{D}_{ba} \right|$

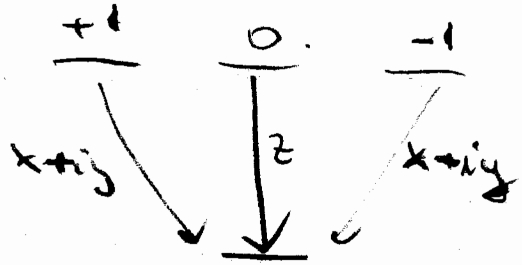
For Case 2: $\Gamma_{ba} = \frac{\omega_0^2 \cdot 2\pi}{\hbar c^2 d} \left| \vec{D}_{ba} \right|^2$

in 3D $\Gamma_{ba} = \frac{4}{3} \frac{\omega_0^3}{\hbar c^3} \left| \vec{D}_{ba} \right|^2$

$\Rightarrow \boxed{\Gamma_{ba}^{2D} = \Gamma_{ba}^{3D} \times \frac{3}{4} \frac{\lambda}{d}} \gg \Gamma_{ba}^{3D}$ for $\lambda \gg d$

c) $\boxed{4}$

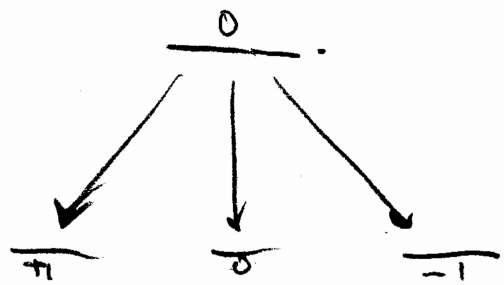
p → s



$$\Gamma_{m=+1} = \Gamma_{m=-1} = 0$$

$$\Gamma^{2D}_{m=0} = \Gamma^{3D}_{m=0} \times \frac{3}{4}\frac{\lambda}{d} \gg \Gamma^{3D}_{ba} \quad \text{for } \lambda \gg d$$

s → p



in 3D: 3 possible transitions

in 2D only one ($\Delta m = 0$)

$$\Rightarrow \Gamma_s^{2D} = \underbrace{\Gamma_s^{3D} \times \frac{3}{4}\frac{\lambda}{d}}_{\text{as above}} \times \frac{1}{3}$$

Belongs to b:

The 2D decay rate is either zero (no modes to carry away the photon) or faster (because of the smaller volume per mode ⇒ 1 photon per mode in the capacitor has a higher electric field in 2D than in 3D).