

## Assignment #10

Due: Wednesday, April 29th, 2009  
TA: Cheng-Hsun Wu / 26-265 / wuch@mit.edu / 617-253-6677  
Office Hours: April 27th (Mon) & April 28th (Tue), 1pm - 3pm.

### 1. The Dressed Atom

In HW assignment 3, problem 2, you looked already at the dressed atom picture using a classical external field. Use the solution you have already derived, and discuss the following properties:

- Identify the a.c. Stark shift in the dressed states. It appears as a shift in the energy levels of the two dressed states relative to the uncoupled states. What is the Stark shift in the limit  $\Omega_1 \ll \delta_L$ . Write the dressed states in this limit.
- The polarization vector is the expectation value of the dipole operator for the dressed state. It can also be written  $\vec{P}(\omega_L, t) = \alpha(\omega_L) E_o \cos(\omega_L t)$ . Compare these two expressions, and express  $\alpha(\omega_L)$  as a function of  $|\langle a | z | b \rangle|^2$ . Substitute the oscillator strength  $f_{ab} = \frac{2m}{\hbar} \omega_o |\langle a | z | b \rangle|^2$  into the expression and compare it with the known expression  $\alpha(\omega_L) = \frac{e^2}{m} \frac{f_{ab}}{\omega_o^2 - \omega_L^2}$ . Where do they agree? Comment briefly on the physics of  $\omega_L \rightarrow \omega_o$ .
- Let  $\omega_L \rightarrow 0$ , and write the d.c. polarizability for both expressions. What approximation has been made that accounts for the difference?
- Let  $|\Psi(t=0)\rangle = |a\rangle$ . Express  $|\Psi(t)\rangle$  as a linear superposition of the dressed states. What is the probability  $P(t)$  that  $|\Psi(t)\rangle$  will be found in the state  $|a\rangle$ ? This shows that the dressed states contain the physics of Rabi "nutation" or oscillation.
- Use your solution to find the stimulated light force on each of the two dressed states.
- Can you explain how the two pictures are connected: (a) the time dependent solutions derived with the classical field (b) the time-independent solutions derived in class using a quantized electromagnetic field.

### 2. Sideband Cooling

As we have seen in class, optical pumping involves using a laser to drive a sequence of atomic transitions into a specific state from which further transitions cannot occur. This method can be used for laser cooling atoms, when the transitions involved are between discrete states of different motional energy (*phonons*). Such is the case with *sideband cooling*, which is essentially optical pumping of an atom which is spatially confined in a harmonic potential; this can be, for example, a trapped ion. Sideband cooling is a very powerful laser cooling technique, which can be used to cool ions to their motional ground state. This problem looks at sideband cooling of a two-level ion in one dimension.

- Consider the Hamiltonian of a two-level atom, with raising and lowering operators  $\sigma_{\pm}$ , interacting with a quantized laser field, through the usual Jaynes-Cummings interaction:

$$H_I = \hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-) \cos(k\hat{x} - \omega t) \quad (1)$$

Suppose that the atom is an ion (of mass  $m$ ) which is confined in a harmonic potential, such that the *Lamb-Dicke parameter*  $\eta = kx_0 \ll 1$ , where  $x_0 = \sqrt{\hbar/(2m\nu)}$  is the size of the ground state wavefunction, and  $\nu$  is the motional frequency of the ion. Expand the interaction Hamiltonian to

first order in  $\eta$  and transform to the frame defined by

$$H_0 = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\nu\hat{a}^\dagger\hat{a}, \quad (2)$$

where  $\hat{\sigma}_+ = |e\rangle\langle g|$  and  $\hat{\sigma}_- = |g\rangle\langle e|$  are the raising and lowering operators,  $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$  is the  $\hat{z}$  axis Pauli matrix, and  $\hat{a}^\dagger$  and  $\hat{a}$  are the motional raising and lowering operators.

What are the Rabi frequencies  $\Omega_{n,n-1}$  and  $\Omega_{n,n+1}$  for transitions  $|n\rangle|g\rangle \rightarrow |n-1\rangle|e\rangle$  (red sideband) and  $|n\rangle|g\rangle \rightarrow |n+1\rangle|e\rangle$  (blue sideband)?

- (b) Because every cooling cycle involves spontaneous emission, coherences decay rapidly and the cooling dynamics are adequately described by rate equations. The population of the harmonic oscillator level  $|n\rangle$ ,  $p_n = \text{Tr}_A \langle n|\rho|n\rangle$  where  $\text{Tr}_A$  denotes the trace over the atomic state, increases due to the spontaneous Raman transitions  $|n-1\rangle|g\rangle \rightarrow |n\rangle|g\rangle$  and  $|n+1\rangle|g\rangle \rightarrow |n\rangle|g\rangle$  and decreases due to the spontaneous Raman transitions  $|n\rangle|g\rangle \rightarrow |n-1\rangle|g\rangle$  and  $|n\rangle|g\rangle \rightarrow |n+1\rangle|g\rangle$ . Note that each spontaneous Raman transition can happen in two ways; for example  $|n\rangle|g\rangle \rightarrow |n+1\rangle|g\rangle$  can proceed via either  $|n\rangle|g\rangle \rightarrow |n\rangle|e\rangle \rightarrow |n+1\rangle|g\rangle$  or  $|n\rangle|g\rangle \rightarrow |n+1\rangle|e\rangle \rightarrow |n+1\rangle|g\rangle$ . Use the Rabi frequencies  $\Omega_{n,n-1}$  and  $\Omega_{n,n+1}$  from part (a) to write down the rate equation for  $p_n$ . Assume for simplicity that the laser power is below saturation ( $\Omega \ll \Gamma$ ), and write your answer in the form

$$\frac{dp_n}{dt} = nA_+p_{n-1} + (n+1)A_-p_{n+1} - (n+1)A_+p_n - nA_-p_n. \quad (3)$$

What are the rates  $A_\pm$ ?

- (c) Write down a differential equation for

$$\langle n \rangle = \sum_{n=0}^{\infty} np_n \quad (4)$$

and solve it. You should find that cooling occurs for  $A_- > A_+$ . What is the steady state temperature in the resolved sideband limit ( $\Gamma \ll \nu$ ) when  $\delta = -\nu$ ? How long does it take to reach this temperature?

- (d) Now for some numbers. What are the steady state temperature and time constant for cooling  $^{88}\text{Sr}$  on the  $S_{1/2} \rightarrow D_{5/2}$  electric quadrupole transition? Use  $\nu = 2\pi \times 10$  MHz,  $\Gamma = 2\pi \times 0.4$  Hz,  $k = 2\pi/(674 \text{ nm})$ , and  $\Omega = 2\pi \times 0.04$  Hz. This does not work experimentally because sideband cooling must compete with heating due to electric field noise, which is typically thousands of quanta per second. To cool in the presence of ion heating it is necessary to use a transition with a linewidth larger than the heating rate. For  $^{88}\text{Sr}$  we broaden the  $D_{5/2}$  state by weakly coupling it to the  $P_{3/2}$  state, which decays quickly via an electric dipole transition to the  $S_{1/2}$  state. What are the steady state temperature and time constant for sideband cooling when  $\Gamma = 2\pi \times 100$  kHz and  $\Omega = 2\pi \times 10$  kHz?