

Solutions to Assignment 10

1. The Dressed Atom

- (a) If we take the $\delta_L \gg \Omega_1$ or $\theta \rightarrow 0$ limit, the original $|b\rangle$ and $|a\rangle$ states are not well coupled by the field, and we are left with $\Psi_+ \rightarrow |b\rangle$ and $\Psi_- \rightarrow |a\rangle$. Thus we should consider the phase shift of $|b\rangle$ relative to Ψ_+ and $|a\rangle$ relative to Ψ_- . This phase shift is $\pm(\omega_L - \Omega)/2$, or an energy difference of $\omega_0 + \delta_L - \Omega$. We know that the unshifted energy difference is ω_0 , so the extra term $\delta_L - \Omega$ must be the Stark shift.

In the $\delta_L \gg \Omega_1$ limit, Ω goes as $\delta_L(1 + \Omega_1^2/2\delta_L^2)$, so the shift is $\Delta_{\pm} = \pm \frac{\Omega_1^2}{4\delta_L}$ for each dressed state.

In this limit, $\cos(\theta) \rightarrow 1$ and $\sin(\theta) \rightarrow \Omega_1/2\delta_L$ and the dressed states become:

$$\begin{aligned}\Psi_+ &= e^{-i(\frac{\omega_0}{2} + \Delta_+)t} \left(e^{-i\omega_L t} \frac{\Omega_1}{2\delta_L} |a\rangle + |b\rangle \right) \\ \Psi_- &= e^{+i(\frac{\omega_0}{2} + \Delta_-)t} \left(|a\rangle - \frac{\Omega_1}{2\delta_L} e^{+i\omega_L t} |b\rangle \right).\end{aligned}$$

Note that this problem illustrates the difficulty of interpreting a time-dependent wave function. Each dressed state has TWO frequency components, which differ by the frequency of the driving laser field. The easiest interpretation is to do spectroscopy and use another laser to drive a transition to a third unperturbed state. Each dressed state will give us two resonances: a strong one (shifted by the AC Stark effect), and a weaker one split of by ω_L . The latter corresponds to a process involving an additional photon of the driving laser.

- (b) We have two expressions for the polarization vector. The dipole operator expectation for the dressed state is (using $\Omega_1 = \frac{eE_0}{\hbar} \langle a|z|b\rangle$)

$$\langle \Psi_+ | ez | \Psi_+ \rangle = \frac{e^2 E_0}{\hbar \delta_L} |\langle a|z|b\rangle|^2 \cos(\omega_L t).$$

Comparing this to the given expression of $\alpha(\omega_L) E_0 \cos(\omega_L t)$ gives $\alpha(\omega_L) = \frac{e^2 |\langle a|z|b\rangle|^2}{\hbar \delta_L}$. Knowing $f_{ab} = \frac{2m}{\hbar} \omega_0 |\langle a|z|b\rangle|^2$, we can express $|\langle a|z|b\rangle|^2 = \frac{\hbar f_{ab}}{2m\omega_0}$, and get

$$\alpha(\omega_L) = \frac{e^2 f_{ab}}{2m\omega_0 \delta_L}.$$

When compared to the known

$$\alpha(\omega_L) = \frac{e^2 f_{ab}}{m(\omega_0^2 - \omega_L^2)}$$

we can see that the approximation $2\omega_0 \simeq \omega_0 + \omega_L$ has been made in our dressed state expression. These two expressions thus agree in the limit $\delta_L \ll \omega_0$. In the dressed atom picture, this approximation was made through the rotating wave approximation. Of course, the dressed atom and the perturbative result can only be compared for small electric fields, as we assumed in (a).

- (c) As $\omega_L \rightarrow 0$, the DC polarization is $E_0 \frac{e^2 f_{ab}}{m\omega_0^2}$. Our expression for the polarization would be only half that. The approximation that was made in our original Hamiltonian was the rotating wave approximation, neglecting the counter-rotating term as small. However, in the DC case, both rotating and counter-rotating contribute equally, explaining the factor of two difference.
- (d) If $|\Psi(t=0)\rangle = |a\rangle$ and $|\Psi(t)\rangle = \alpha_+ |\Psi_+(t)\rangle + \alpha_- |\Psi_-(t)\rangle$ we find $\alpha_+ = \sin(\theta)$ and $\alpha_- = \cos(\theta)$ give us the correct behavior at $t=0$. Then (after some trig even I can do) $|\langle a|\Psi(t)\rangle|^2 = 1 - \sin^2(2\theta) \sin^2(\Omega t/2) = 1 - \frac{\Omega_1^2}{\Omega^2} \sin^2(\Omega t/2)$. This is the Rabi oscillation of $|a\rangle$ in laser field.
- (e) From part (b) one can write down the time dependent potential of dressed states Ψ_{\pm} .

$$U_{\pm}(t) = -\frac{1}{2} \vec{P}_{\pm}(t) \vec{E}(t) = \frac{\mp \hbar \Omega_1^2}{2\delta_L} \cos^2(\omega_L t)$$

Taking the time average, we have $\overline{U_{\pm}(t)} = \mp \frac{\hbar \Omega_1^2}{4\delta_L}$. So the force is $\pm \frac{\hbar \nabla \Omega_1^2}{4\delta_L}$.

Alternatively, one could calculate the expectation value for the total energy, i.e. evaluate the expectation value for the operator $i\hbar \frac{d}{dt}$. For the state Ψ_+ , we obtain $\frac{\hbar \Omega}{2} + \hbar \sin^2 \theta \omega_L$. Naively, the force would be the derivative of it, which includes the derivative of the second term. However, this is incorrect, since it corresponds to the excited state which has absorbed a photon from the laser field. In other words, we are including here only the energy of the atom, but not of the laser field. If we would include the laser field, then the photon energy would not appear. We avoided this difficulty, when we simply calculated the force of the external laser field onto the induced atomic dipole moment.

- (f) In class, we derived the forces seen by the upper and lower levels Ψ_{\pm} are $F_{\pm} = \mp \frac{1}{2} \hbar \nabla \Omega$. In the $\delta_L \gg \Omega_1$ limit, we have

$$F_{\pm} = \mp \frac{1}{2} \hbar \nabla \sqrt{\Omega_1^2 + \delta_L^2} = \mp \frac{\nabla \Omega_1^2}{4\sqrt{\Omega_1^2 + \delta_L^2}} \approx \pm \frac{\hbar \nabla \Omega_1^2}{4\delta_L}$$

Comparing this to part (e), we conclude that the two pictures agree when we use the time average of the force exerted by the classical laser field.

Note that the treatment in class with the quantized treatment of the electromagnetic field avoided the use of time dependent fields and Hamiltonians by assuming the em. field to be a Fock state with a given photon number. A classical field is a coherent state with the same mean photon number, but we have a time dependence due to the beat note of the different Fock states which are superimposed to give the coherent state.

You may wonder why one treatment (using the external potential $-\frac{1}{2}\vec{P}\vec{E}$) gives a time dependence at a frequency of $2\omega_L$, why others don't (using the approach with quantized EM field, or simply using the derivative of the time-dependent wave function). I am not 100 % sure, but the answer should be related to the rotating wave approximation, where terms rapidly oscillating at $2\omega_L$ are discarded. When we calculated $-\frac{1}{2}\vec{P}\vec{E}$, we used a solution for \vec{P} which was obtained within the RWA, but multiplied it with the total field E , which has co- and counter-rotating parts.

2. Sideband Cooling

(a) Expanding to first order in η , we get

$$\begin{aligned} H_I &= \frac{\hbar\Omega}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)\left(e^{i(k\hat{x}-\omega t)} + e^{-i(k\hat{x}-\omega t)}\right) \\ &= \frac{\hbar\Omega}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)\left([1 + i\eta(\hat{a} + \hat{a}^\dagger)]e^{-i\omega t} + [1 - i\eta(\hat{a} + \hat{a}^\dagger)]e^{i\omega t}\right). \end{aligned}$$

The rotating frame Hamiltonian, H'_I , is given by the transformation $H_I \rightarrow H'_I = e^{iH_0t/\hbar}H_Ie^{-iH_0t/\hbar}$. This is equivalent to transforming the operators $\hat{\sigma}_\pm$, \hat{a} , and \hat{a}^\dagger according to $\hat{\sigma}_\pm \rightarrow e^{iH_0t/\hbar}\hat{\sigma}_\pm e^{-iH_0t/\hbar} = \hat{\sigma}_\pm e^{\pm i\omega_0 t}$, $\hat{a} \rightarrow e^{iH_0t/\hbar}\hat{a}e^{-iH_0t/\hbar} = \hat{a}e^{-i\nu t}$, and $\hat{a}^\dagger \rightarrow e^{iH_0t/\hbar}\hat{a}^\dagger e^{-iH_0t/\hbar} = \hat{a}^\dagger e^{i\nu t}$, so we get

$$\begin{aligned} H_I &= \frac{\hbar\Omega}{2}(\hat{\sigma}_+e^{i\omega_0 t} + \hat{\sigma}_-e^{-i\omega_0 t}) \\ &\quad \times \left([1 + i\eta(\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t})]e^{-i\omega t} + [1 - i\eta(\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t})]e^{i\omega t}\right). \end{aligned}$$

Finally, throwing away the terms rotating at $\omega_0 + \omega$ and $\omega_0 + \omega \pm \nu$ we get

$$\begin{aligned} H_I &= \frac{\hbar\Omega}{2}\left(\hat{\sigma}_+e^{-i\delta t} + \hat{\sigma}_-e^{i\delta t}\right) \\ &\quad + i\frac{\hbar\Omega}{2}\eta\left(\hat{\sigma}_+\hat{a}e^{-i(\delta+\nu)t} - \hat{\sigma}_-\hat{a}^\dagger e^{i(\delta+\nu)t}\right) \\ &\quad + i\frac{\hbar\Omega}{2}\eta\left(\hat{\sigma}_+\hat{a}^\dagger e^{-i(\delta-\nu)t} - \hat{\sigma}_-\hat{a}e^{i(\delta-\nu)t}\right). \end{aligned}$$

We can read off the Rabi frequencies using

$$\frac{\hbar\Omega_{n,n\pm 1}}{2} = \langle n \pm 1 | \langle e | H'_I | n \rangle | g \rangle$$

to be $\Omega_{n,n-1} = \Omega\sqrt{n}\eta$ and $\Omega_{n,n+1} = \Omega\sqrt{n+1}\eta$.

(b) This part is done in the class notes. The rates are given by

$$A_\pm = \frac{\Omega^2\eta^2}{\Gamma}[\alpha W(\delta) + W(\delta \mp \nu)]$$

where

$$W(\delta) = \frac{1}{1 + \left(\frac{2\delta}{\Gamma}\right)^2}.$$

(c) The first half of this part is done in the class notes:

$$\frac{d}{dt}\langle n \rangle = -(A_- - A_+)\langle n \rangle + A_+$$

with solution

$$\langle n \rangle_t = \langle n \rangle_{t=0} e^{-(A_- - A_+)t} + \langle n \rangle_{SS} [1 - e^{-(A_- - A_+)t}]$$

where

$$\langle n \rangle_{SS} = \frac{A_+}{A_- - A_+} .$$

From here we find that the steady state temperature is

$$T_{SS} = \frac{\hbar\nu}{k_B} \frac{1}{\ln\left(\frac{A_-}{A_+}\right)} = \frac{\hbar\nu}{k_B} \frac{1}{\ln\left(\left(\frac{2\nu}{\Gamma}\right)^2 \frac{4}{1+4\alpha}\right)}$$

and the time constant is

$$\tau = \frac{1}{A_- - A_+} = \frac{\Gamma}{\Omega^2 \eta^2} .$$

(d) Plugging in the numbers and using $\alpha = 2/7$ for an electric quadrupole transition we get $T_{SS} = 13 \mu\text{K}$ and $\tau = 8.0 \times 10^4 \text{ s}$ for cooling on the natural $S_{1/2} \rightarrow D_{5/2}$ transition, and $T_{SS} = 42 \mu\text{K}$ and $\tau = 0.32 \text{ s}$ for cooling on the broadened transition.