MIT 8.422 Atomic Physics II *Spring, 2009*
Prof. Wolfgang Ketterle February 18, 2009
Prof. Isaac Chuang

Assignment #2

Due: Wednesday, February 25, 2009
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Office Hours: Feb 23rd (Mon) & Feb 24th (Tue), 4pm - 6pm.

1. **The quantum beamsplitter**. Let the beamsplitter operator B, acting with angle θ on modes a and b, be defined by

$$B = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right] \; . \tag{1}$$

In lecture, we noted that B conserves total photon number, and leaves coherent states as coherent states. In this problem, you can prove those claims for yourself.

a) Prove that B leaves $n_a + n_b = a^\dagger a + b^\dagger b$ unchanged. Also prove that $B^\dagger B = I$.

b) Let $|\alpha\rangle$ be a coherent state. Compute $B|0\rangle_b|\alpha\rangle_a$, and show that the output is a tensor product of coherent states for all θ. Your result should be consistent with the intuition that the beamsplitter has well defined transmission and reflection coefficients; give these as a function of θ.

c) There is close connection between the Lie group $SU(2)$ and the algebra of two coupled harmonic oscillators, which is useful for understanding B. Show that if we define

$$s_z = a^\dagger a - b^\dagger b \qquad s_+ = a^\dagger b \qquad s_- = ab^\dagger \; , \tag{2}$$

and let $s_\pm = (s_x \pm i s_y)/2$, then s_x, s_y, and s_z have the same commutation relations as the Pauli matrices. This relationship also explains why $a^\dagger a + b^\dagger b$ is invariant; it is the Casimir operator of the algebra.

d) How does a beamsplitter transform an input photon-number eigenstate? Let

$$B(\theta) = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right] \; , \tag{3}$$

and $B = B(\pi/4)$ be a 50/50 beamsplitter, such that

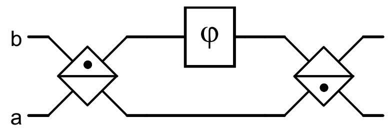
$$BaB^\dagger = \frac{a+b}{\sqrt{2}} \qquad \text{and} \qquad BbB^\dagger = \frac{-a+b}{\sqrt{2}} \; . \tag{4}$$

Compute $B|0\rangle|n\rangle$, where the first label is mode b, and the second label is mode a. Note that the result is *not* $|n/2\rangle|n/2\rangle$, because $|n\rangle$ is a photon number eigenstate, and not a coherent state. Hint: use the binomial expansion on $(a^\dagger + b^\dagger)^n$.

2. **Heisenberg-limited interferometry with the Yurke state**

The Yurke state $|\psi\rangle = (|n\rangle|n-1\rangle + |n-1\rangle|n\rangle)/\sqrt{2}$ allows one to obtain a measurement of an unknown phase ϕ with uncertainty $\langle\Delta\phi\rangle = \frac{\sqrt{2}}{n}$, using a Mach-Zehnder interferometer. Methods for experimentally realizing these states have been proposed, for example, using Bose-Einstein condenstates [Castin & Dalibard, Phys. Rev. A vol. 55, p. 4330, 1997].

a) Let us now analyze the Mach-Zehnder interferometer fed with a Yurke state as input. Use this setup:



and work in the Schrodinger picture, by doing the following. Let the input be the Yurke state, $|\phi_0\rangle = |\psi\rangle$, let the state after the first 50/50 beamsplitter be $|\phi_1\rangle = B|\phi_0\rangle$, the state after the phase shifter be $|\phi_2\rangle = P|\phi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\phi_3\rangle = B^\dagger|\phi_2\rangle$. Give expressions for $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$. Note that the transform of the phase shifter P is $PaP^\dagger = ae^{i\phi}$. Double-check that when $\phi = 0$, the output is the same as the input $|\phi_3\rangle = |\phi_0\rangle$. Hint: write these states in terms of operators acting on the vacuum.

b) What is the uncertainty with which you can determine ϕ using the Yurke state input? This is

$$\langle \Delta\phi^2 \rangle = \frac{\langle \Delta M^2 \rangle}{\left| \frac{\partial \langle M \rangle}{\partial \phi} \right|^2}\,, \tag{5}$$

where $M = a^\dagger a - b^\dagger b$ is the difference in the photon numbers measured at the outputs of the interferometer. Compute $\langle \Delta\phi^2 \rangle$, evaluated at $\phi = 0$ (the point at which the interferometer is balanced), using the $|\phi_3\rangle$ you obtained above. You should find $\langle \Delta\phi \rangle = \frac{\sqrt{2}}{n}$.

c) In lecture, we used the Heisenberg picture to compute statistics about interferometer performance with coherent state inputs. For comparison with the Yurke state, let's now work out what happens with coherent states in the Schrodinger picture. Using the same diagram as above, let the input now be a coherent state and a vacuum state, $|\psi_0\rangle = |\alpha\rangle|0\rangle$. Just as above, let the state after the first 50/50 beamsplitter be $|\psi_1\rangle = B|\psi_0\rangle$, the state after the phase shifter be $|\psi_2\rangle = P|\psi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\psi_3\rangle = B^\dagger|\psi_2\rangle$. Give expressions for $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$.

d) Given the quantum fluctuations of the coherent state, use uncertainty propagation to determine the uncertainty with which you can determine ϕ using the coherent state input, as a function of $\bar{n} = |\alpha|^2$ and the phase shift angle ϕ.

3. **The Schmidt measure of pure state entanglement**

Entanglement is a property of a composite quantum system that cannot be changed by local operations and classical communications. How do we mathematically determine if a given state is entangled or not? And if a state is entangled, how entangled is it?

In this problem, we explore a measure of bi-partite entanglement known as the *Schmidt number*, which is particularly easy to compute. This is based on the Schmidt decomposition, which, for a pure state $|\psi\rangle$ in the Hilbert space of systems A and B, is the expression of $|\psi\rangle$ in the form

$$|\psi\rangle = \sum_k \lambda_k |k_A\rangle |k_B\rangle\,, \tag{6}$$

where $|k_A\rangle$ and $|k_B\rangle$ are orthonormal states of systems A and B, respectively, and $\sum_k \lambda_k^2 = 1$. Note that this is essentially just a singular value decomposition. The Schmidt number is defined as the number of nonzero λ_k.

a) Prove that $|\psi\rangle$ is a product state, that is $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$, for some states $|\psi_A\rangle$ and $|\psi_B\rangle$ of systems A and B, if and only if the Schmidt number of $|\psi\rangle$ is 1.

b) Prove that the Schmidt number cannot be changed by local unitary transforms. It turns out that even with any amount of additional classical communication, the Schmidt number still cannot be changed. Because this number is invariant under "local operations and classical communication," it is a measure of entanglement.

c) Give the Schmidt numbers for each of the following states:

$$|\phi_1\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}} \tag{7}$$

$$|\phi_2\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \tag{8}$$

$$|\phi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \tag{9}$$

$$|\phi_4\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} . \tag{10}$$

d) From the lecture, it would seem to follow that the more entangled a state you have (eg the higher Schmidt number a state has), the better you can do with interferometric measurements.

What is the Schmidt number of the Yurke state, $|\psi\rangle = (|n\rangle|n-1\rangle + |n-1\rangle|n\rangle)/\sqrt{2}$? Explain why this is the wrong state to compute the entanglement of. What is the Schmidt number of the state after the Yurke state is transformed by a 50/50 beamsplitter B, $|\phi_1\rangle = B|\psi\rangle$? Explain why this is a better measure.

What is the Schmidt number of the state given by feeding a coherent state $|\alpha\rangle$ and vacuum $|0\rangle$ into B, i.e. $|\psi_1\rangle = B|\alpha\rangle|0\rangle$?

e) Often, the Schmidt number can lack meaning as a quantitative measure of entanglement, because it includes even the smallest non-zero coefficients in its count. One improvement on this is to count only coefficients above a certain threshold, as illustrated by the following example.

A two-mode squeezed state can be generated in the laboratory by a certain kind of optical parameteric oscillator; suppose this is the state

$$|\Psi\rangle = \exp\left[-\frac{r}{2}(a_1 a_2 - a_1^\dagger a_2^\dagger)\right]|0\rangle_1|0\rangle_2 \propto \frac{1}{\cosh r}\sum_n \tanh^n r |n\rangle_1|n\rangle_2 . \tag{11}$$

Plot the number of Schmidt coefficients of $|\Psi\rangle$ which are above a fixed threshold, say 0.01, as a function of the squeezing parameter r. Show that this measure of entanglement increases with increasing r, as intuitively desired.

The concept behind this truncation method leads to the truncations used in efficient classical simulation of coupled spin systems, with the density matrix renormalization group and matrix product state techniques.