

Assignment #2 Solutions

Due: Wednesday, February 25, 2009
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 Office Hours: Feb 23rd (Mon) & Feb 24th (Tue), 4pm - 6pm.

1. **The quantum beamsplitter.** Let the beamsplitter operator B , acting with angle θ on modes a and b , be defined by

$$B = \exp [\theta (a^\dagger b - ab^\dagger)] . \quad (1)$$

In lecture, we noted that B conserves total photon number, and leaves coherent states as coherent states. In this problem, you can prove those claims for yourself.

- a) Prove that B leaves $n_a + n_b = a^\dagger a + b^\dagger b$ unchanged. Also prove that $B^\dagger B = I$.

We'd like to show that $(a^\dagger a + b^\dagger b)$ passes through $e^{\theta(a^\dagger b - ab^\dagger)} = \sum_n \theta \frac{(a^\dagger b - ab^\dagger)^n}{n!}$. It is sufficient to show that $[a^\dagger b - b^\dagger a, n_a + n_b] = 0$:

$$[a^\dagger b - b^\dagger a, n_a + n_b] = [a^\dagger b - b^\dagger a, a^\dagger a - b^\dagger b] \quad (2)$$

$$= [a^\dagger b, a^\dagger a] - [b^\dagger a, a^\dagger a] + [a^\dagger b, b^\dagger b] - [b^\dagger a, b^\dagger b] \quad (3)$$

$$= -ba^\dagger - b^\dagger a + a^\dagger b + ab^\dagger \quad (4)$$

$$= 0. \quad (5)$$

Consequently,

$$(n_a + n_b)B = \sum_n \theta (a^\dagger a + b^\dagger b) \frac{(a^\dagger b - ab^\dagger)^n}{n!} \quad (6)$$

$$= \sum_n \theta \frac{(a^\dagger b - ab^\dagger)^n}{n!} (a^\dagger a + b^\dagger b) = B(n_a + n_b) \quad (7)$$

Next, to prove that $B^\dagger B = I$, observe that $B = \exp[iH]$, where $H = -i\theta(a^\dagger b - b^\dagger a)$ is Hermitian. Thus, $B^\dagger = \exp[-iH]$. We can then write:

$$B^\dagger B = \exp[iH] * \exp[-iH] \quad (8)$$

$$= \exp[iH - iH] \quad (\text{Because } [H, H] = 0) \quad (9)$$

$$= I \quad (10)$$

$$(11)$$

One could also note that using the BCH as stated in the notes (2.3.1) we have the case of $A = 1, \Lambda = \theta, G = (ab^\dagger - a^\dagger b)$. This means that $c_0 = 1$ and all other terms are 0. That leaves us with I by inspection.

- b) Let $|\alpha\rangle$ be a coherent state. Compute $B|0\rangle_b|\alpha\rangle_a$, and show that the output is a tensor product of coherent states for all θ . Your result should be consistent with the intuition that the beamsplitter has well defined transmission and reflection coefficients; give these as a function of θ .

This one can be messy to work out in summation form, but if we recall that coherent states are eigenfunctions of the a, b operators and that $B^\dagger B = I$, we can simplify matters greatly.

$$aB|0\rangle_b|\alpha\rangle_a = BB^\dagger aB|0\rangle_b|\alpha\rangle_a \quad (12)$$

$$= B(a \cos \theta - b \sin \theta)|0\rangle_b|\alpha\rangle_a \quad (13)$$

$$= \alpha \cos \theta B|0\rangle_b|\alpha\rangle_a \quad (14)$$

This implies that $B|0\rangle_b|\alpha\rangle_a$ is an eigenstate of a with eigenvalue $\alpha \cos \theta$. A similar calculation for b shows that $B|0\rangle_b|\alpha\rangle_a$ is an eigenstate of b with eigenvalue $\alpha \sin \theta$. Therefore, up to a global phase, $B|0\rangle_b|\alpha\rangle_a$ is equal to the coherent state tensor product $|\alpha \sin \theta\rangle_b|\alpha \cos \theta\rangle_a$. The beamsplitter looks like it has $R = r^2 = \sin^2(\theta)$ and $T = t^2 = \cos^2(\theta)$.

- c) There is close connection between the Lie group $SU(2)$ and the algebra of two coupled harmonic oscillators, which is useful for understanding B . Show that if we define

$$s_z = a^\dagger a - b^\dagger b \quad s_+ = a^\dagger b \quad s_- = ab^\dagger, \quad (15)$$

and let $s_\pm = (s_x \pm i s_y)/2$, then s_x, s_y , and s_z have the same commutation relations as the Pauli matrices. This relationship also explains why $a^\dagger a + b^\dagger b$ is invariant; it is the Casimir operator of the algebra.

From the definitions of s_+ and s_- we can solve for s_x and s_y :

$$s_x = a^\dagger b + ab^\dagger \quad (16)$$

$$s_y = i(ab^\dagger - a^\dagger b) \quad (17)$$

$$(18)$$

Using the given value for $s_z = a^\dagger a - b^\dagger b$, we can now directly evaluate the commutators:

$$[s_x, s_y] = [a^\dagger b + ab^\dagger, i(ab^\dagger - a^\dagger b)] = 2ia^\dagger a - 2ib^\dagger b = 2is_z \quad (19)$$

$$[s_y, s_z] = [i(ab^\dagger - a^\dagger b), a^\dagger a - b^\dagger b] = 2ia^\dagger b + 2iab^\dagger = 2is_x \quad (20)$$

$$[s_z, s_x] = [a^\dagger a - b^\dagger b, a^\dagger b + ab^\dagger] = 2a^\dagger b - 2ab^\dagger = 2is_y \quad (21)$$

$$(22)$$

These are the same commutation relations as the Pauli matrices.

d) How does a beamsplitter transform an input photon-number eigenstate? Let

$$B(\theta) = \exp[\theta(a^\dagger b - ab^\dagger)] , \quad (23)$$

and $B = B(\pi/4)$ be a 50/50 beamsplitter, such that

$$BaB^\dagger = \frac{a+b}{\sqrt{2}} \quad \text{and} \quad BbB^\dagger = \frac{-a+b}{\sqrt{2}} . \quad (24)$$

Compute $B|0\rangle|n\rangle$, where the first label is mode b , and the second label is mode a . Note that the result is *not* $|n/2\rangle|n/2\rangle$, because $|n\rangle$ is a photon number eigenstate, and not a coherent state. Hint: use the binomial expansion on $(a^\dagger + b^\dagger)^n$.

Making heavy use of the fact that $B^\dagger B = I$:

$$B|0\rangle|n\rangle = B \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle|0\rangle \quad (25)$$

$$= \frac{(Ba^\dagger B^\dagger)^n}{\sqrt{n!}} |0\rangle|0\rangle \quad (26)$$

$$= \frac{1}{\sqrt{n!}} \left(\frac{a^\dagger + b^\dagger}{\sqrt{2}} \right)^n |0\rangle|0\rangle \quad (27)$$

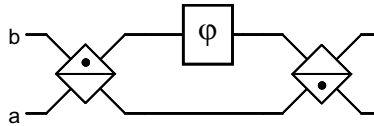
$$= \frac{1}{\sqrt{2^n n!}} \sum_{k=0}^n \binom{n}{k} a^{\dagger k} (b^\dagger)^{n-k} |0\rangle|0\rangle \quad (28)$$

$$= \sum_{k=0}^n \sqrt{\frac{n!}{2^n k!(n-k)!}} |n-k\rangle|k\rangle . \quad (29)$$

2. Heisenberg-limited interferometry with the Yurke state

The Yurke state $|\psi\rangle = (|n\rangle|n-1\rangle + |n-1\rangle|n\rangle)/\sqrt{2}$ allows one to obtain a measurement of an unknown phase ϕ with uncertainty $\langle \Delta\phi \rangle = \frac{\sqrt{2}}{n}$, using a Mach-Zehnder interferometer. Methods for experimentally realizing these states have been proposed, for example, using Bose-Einstein condensates [Castin & Dalibard, Phys. Rev. A vol. 55, p. 4330, 1997].

a) Let us now analyze the Mach-Zehnder interferometer fed with a Yurke state as input. Use this setup:



and work in the Schrodinger picture, by doing the following. Let the input be the Yurke state, $|\phi_0\rangle = |\psi\rangle$, let the state after the first 50/50 beamsplitter be $|\phi_1\rangle = B|\phi_0\rangle$, the state after the phase shifter be $|\phi_2\rangle = P|\phi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\phi_3\rangle = B^\dagger|\phi_2\rangle$. Give expressions for $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$. Note that the transform of the phase shifter P is $PaP^\dagger = ae^{i\phi}$. Double-check that when $\phi = 0$, the output is the same as the input $|\phi_3\rangle = |\phi_0\rangle$. Hint: write these states in terms of operators acting on the vacuum.

Here it makes things simpler to write the Yurke state as

$$|\phi_0\rangle = \frac{1}{\sqrt{2n!(n-1)!}}(a^\dagger + b^\dagger)(a^\dagger b^\dagger)^{n-1} |0\rangle |0\rangle .$$

Making use again of $B^\dagger B = I$, we get

$$|\phi_1\rangle = B |\phi_0\rangle \quad (30)$$

$$= \frac{(-i)^{n-1}(1-i)}{2^n \sqrt{n!(n-1)!}}(a^\dagger + b^\dagger)(a^{\dagger 2} + b^{\dagger 2})^{n-1} |0\rangle |0\rangle , \quad (31)$$

$$|\phi_2\rangle = P |\phi_1\rangle \quad (32)$$

$$= \frac{(-i)^{n-1}(1-i)}{2^n \sqrt{n!(n-1)!}}(a^\dagger e^{-i\phi} + b^\dagger)(a^{\dagger 2} e^{-2i\phi} + b^{\dagger 2})^{n-1} |0\rangle |0\rangle , \quad (33)$$

and

$$|\phi_3\rangle = B^\dagger |\phi_2\rangle \quad (34)$$

$$= \frac{(-i)^{n-1}(1-i)}{2^{2n-1} \sqrt{2n!(n-1)!}} [(a^\dagger + ib^\dagger)e^{-i\phi} + (b^\dagger + ia^\dagger)] [(a^\dagger + ib^\dagger)^2 e^{-2i\phi} + (b^\dagger + ia^\dagger)^2]^{n-1} |0\rangle |0\rangle \quad (35)$$

When $\phi = 0$, we get $|\phi_3\rangle = |\phi_0\rangle$ as expected.

- b) What is the uncertainty with which you can determine ϕ using the Yurke state input? This is

$$\langle \Delta\phi^2 \rangle = \frac{\langle \Delta M^2 \rangle}{\left| \frac{\partial \langle M \rangle}{\partial \phi} \right|^2}, \quad (36)$$

where $M = a^\dagger a - b^\dagger b$ is the difference in the photon numbers measured at the outputs of the interferometer. Compute $\langle \Delta\phi^2 \rangle$, evaluated at $\phi = 0$ (the point at which the interferometer is balanced), using the $|\phi_3\rangle$ you obtained above. You should find $\langle \Delta\phi \rangle = \frac{\sqrt{2}}{n}$.

Since the interferometer is balanced at $\phi = 0$, we can use $|\phi_3\rangle = |\phi_0\rangle$ to calculate $\langle \Delta M^2 \rangle$:

$$\langle M \rangle = \langle \phi_0 | (a^\dagger a - b^\dagger b) | \phi_0 \rangle = 0$$

and

$$\langle M^2 \rangle = \langle \phi_0 | (a^\dagger a - b^\dagger b)^2 | \phi_0 \rangle = 1 ,$$

so

$$\langle \Delta M^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2 = 1 .$$

Remembering the action of derivatives on expectation values, the derivative of the output with respect to ϕ can be written as

$$\frac{\partial \langle M \rangle}{\partial \phi} = \left(\frac{\partial}{\partial \phi} \langle \phi_3 | \right) M | \phi_3 \rangle \Big|_{\phi=0} + \langle \phi_3 | M \left(\frac{\partial}{\partial \phi} | \phi_3 \rangle \right) \Big|_{\phi=0} \quad (37)$$

$$= \left(\frac{\partial}{\partial \phi} \langle \phi_3 | \Big|_{\phi=0} \right) M | \phi_0 \rangle + \langle \phi_0 | M \left(\frac{\partial}{\partial \phi} | \phi_3 \rangle \Big|_{\phi=0} \right) . \quad (38)$$

Using

$$\begin{aligned} \frac{\partial}{\partial \phi} | \phi_3 \rangle \Big|_{\phi=0} &= \frac{1}{2\sqrt{2}} [((n-1)(1-2i) + (1-i)) |n\rangle |n-1\rangle - ((n-1)(1+2i) + (1+i)) |n-1\rangle |n\rangle] \\ &\quad + \text{terms orthogonal to } | \phi_0 \rangle \quad (39) \end{aligned}$$

we get

$$\frac{\delta \langle M \rangle}{\delta \phi} = -n .$$

Putting everything together, we find that

$$\sqrt{\langle \Delta \phi^2 \rangle} = \frac{\sqrt{\langle \Delta M^2 \rangle}}{\left| \frac{\partial \langle M \rangle}{\partial \phi} \right|} = \frac{1}{n} .$$

- c) In lecture, we used the Heisenberg picture to compute statistics about interferometer performance with coherent state inputs. For comparison with the Yurke state, let's now work out what happens with coherent states in the Schrodinger picture. Using the same diagram as above, let the input now be a coherent state and a vacuum state, $|\psi_0\rangle = |\alpha\rangle|0\rangle$. Just as above, let the state after the first 50/50 beamsplitter be $|\psi_1\rangle = B|\psi_0\rangle$, the state after the phase shifter be $|\psi_2\rangle = P|\psi_1\rangle$, and the state after the final 50/50 beamsplitter be $|\psi_3\rangle = B^\dagger|\psi_2\rangle$. Give expressions for $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$. Making use our work in Problem 1, we get

$$|\phi_1\rangle = B|\alpha\rangle_a|0\rangle_b \quad (41)$$

$$= |\alpha/\sqrt{2}\rangle|\alpha/\sqrt{2}\rangle , \quad (42)$$

$$|\phi_2\rangle = P|\phi_1\rangle \quad (43)$$

$$= |(\alpha e^{i\phi})/\sqrt{2}\rangle|\alpha/\sqrt{2}\rangle , \quad (44)$$

and

$$|\phi_3\rangle = B^\dagger|\phi_2\rangle \quad (45)$$

$$= |\alpha e^{i\phi/2} \cos(\phi/2)\rangle - i\alpha e^{i\phi/2} \sin(\phi/2)\rangle . \quad (46)$$

- d) Given the quantum fluctuations of the coherent state, use uncertainty propagation to determine the uncertainty with which you can determine ϕ using the coherent state input, as a function of $\bar{n} = |\alpha|^2$ and the phase shift angle ϕ .

Let's try to work out $\langle X \rangle$ and $\langle X^2 \rangle$ given our result from c):

$$\langle X \rangle = \langle (a^\dagger a - b^\dagger b) \rangle = n \cos^2(\phi/2) - n \sin^2(\phi/2) = n \cos(\phi) \quad (47)$$

$$\langle X^2 \rangle = \langle (a^\dagger(a^\dagger a + 1)a + b^\dagger(b^\dagger b + 1)b - 2a^\dagger a b^\dagger b) \rangle \quad (48)$$

$$= n^2 \cos^2(\phi) + n(\cos(\phi/2) + \sin(\phi/2)) \quad (49)$$

We can easily evaluate $\partial\langle X \rangle/\partial\phi$, giving us:

$$\langle \Delta\phi^2 \rangle = \frac{\langle X^2 \rangle - \langle X \rangle^2}{\left| \frac{\partial\langle X \rangle}{\partial\phi} \right|^2} = \frac{1}{\bar{n}} \frac{\cos(\phi/2) + \sin(\phi/2)}{\sin^2(\phi)} \quad (50)$$

3. The Schmidt measure of pure state entanglement

Entanglement is a property of a composite quantum system that cannot be changed by local operations and classical communications. How do we mathematically determine if a given state is entangled or not? And if a state is entangled, how entangled is it?

In this problem, we explore a measure of bi-partite entanglement known as the *Schmidt number*, which is particularly easy to compute. This is based on the Schmidt decomposition, which, for a pure state $|\psi\rangle$ in the Hilbert space of systems A and B , is the expression of $|\psi\rangle$ in the form

$$|\psi\rangle = \sum_k \lambda_k |k_A\rangle |k_B\rangle, \quad (51)$$

where $|k_A\rangle$ and $|k_B\rangle$ are orthonormal states of systems A and B , respectively, and $\sum_k \lambda_k^2 = 1$. Note that this is essentially just a singular value decomposition. The Schmidt number is defined as the number of nonzero λ_k .

- a) Prove that $|\psi\rangle$ is a product state, that is $|\psi\rangle = |\psi_A\rangle |\psi_B\rangle$, for some states $|\psi_A\rangle$ and $|\psi_B\rangle$ of systems A and B , if and only if the Schmidt number of $|\psi\rangle$ is 1.

From the definition of the Schmidt number, it falls directly that Schmidt number of a product state is 1. The converse is easier to prove if we use the definition of the Schmidt number $Sch(|\psi\rangle) = \text{Rank}(\text{Tr}_B(\rho))$ where Tr_B means the trace over the eigenstates of B , $|k_B\rangle$, as given in Nielsen and Chuang (2005) for instance. If $\text{Rank}(\text{Tr}_B(\rho)) = 1$, we know that ρ must be the direct product $\rho_A \otimes \rho_B$, which is the density matrix of a product state. We could also use the fact that $\text{Rank}(A + B) \leq \text{Rank}(A) + \text{Rank}(B)$.

- b) Prove that the Schmidt number cannot be changed by local unitary transforms. It turns out that even with any amount of additional classical communication, the Schmidt number still cannot be changed. Because this number is invariant under "local operations and classical communication," it is a measure of entanglement.

If we transform ρ_B by a unitary transform U , the Schmidt number becomes

$\text{Rank}(\text{Tr}(U^\dagger \rho_B U)) = \text{Rank}(\text{Tr}(\rho_B)) = \text{Sch}(|\psi\rangle)$ as $\text{Tr}(U^\dagger A U) = \text{Tr}(A)$.

c) Give the Schmidt numbers for each of the following states:

$$|\phi_1\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}} \quad (52)$$

$$|\phi_2\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \quad (53)$$

$$|\phi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \quad (54)$$

$$|\phi_4\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}. \quad (55)$$

By inspection, $|\phi_1\rangle$ is a sum of $|00\rangle + |11\rangle + |22\rangle$ with non-zero eigenvalues. So, $\text{Sch}(|\phi_1\rangle) = 3$.

We note $|\phi_2\rangle$ is the direct product of $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ with $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, so $\text{Sch}(|\phi_2\rangle) = 1$.

If we write out the density matrix

$$\rho_3 \approx \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \quad (56)$$

where I've used the basis ordering $|00\rangle |01\rangle |10\rangle |11\rangle$, we find that the Trace over B leaves us with

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (57)$$

which is a rank 2 matrix. Therefore, $\text{Sch}(|\phi_3\rangle) = 2$.

If we write out the density matrix

$$\rho_4 \approx \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad (58)$$

where I've used the same basis ordering as above, we find that the Trace over B leaves us with

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad (59)$$

which is a rank 2 matrix. Therefore, $\text{Sch}(|\phi_4\rangle) = 2$.

d) From the lecture, it would seem to follow that the more entangled a state you have (eg the higher Schmidt number a state has), the better you can do with interferometric measurements.

What is the Schmidt number of the Yurke state, $|\psi\rangle = (|n\rangle|n-1\rangle + |n-1\rangle|n\rangle)/\sqrt{2}$? Explain why this is the wrong state to compute the entanglement of. What is the Schmidt number of the state after the Yurke state is transformed by a 50/50 beamsplitter B , $|\phi_1\rangle = B|\psi\rangle$? Explain why this is a better measure.

The initial Yurke state has Schmidt number 2 by inspection—just use the two product states of the Yurke state as two of the $|k_A\rangle|k_B\rangle$. However, this is not the state we apply the phase shift to. Consequently, we do better to use $|\phi_1\rangle = B|\psi\rangle$, the state that the phase shifter acts upon as our measure. Note, I'm going to use the definition of the beamsplitter action from problem 1 and apply the binomial expansion:

$$|\phi_1\rangle = B|\phi_0\rangle \quad (60)$$

$$= \frac{1}{\sqrt{n!(n-1)!}} (b^\dagger) \left(\frac{a^\dagger + b^\dagger}{2} \right)^{n-1} |0\rangle|0\rangle \quad (61)$$

$$= \sum_k^{n-1} \frac{1}{\sqrt{n!(n-1)!}} C_k^{n-1} (a^\dagger)^{2k} (b^\dagger)^{2n-2k-1} |0\rangle|0\rangle \quad (62)$$

$$= \sum_k^{n-1} g_k |2k\rangle|2n-2k-1\rangle \quad (63)$$

where g_k is a non-zero coefficient. Now, if we write out the density matrix for this state, we obtain:

$$= \sum_{k,j}^{n-1,m-1} g_k g_j^* |2k\rangle|2n-2k-1\rangle \langle 2m-2l-1| \langle 2l|. \quad (64)$$

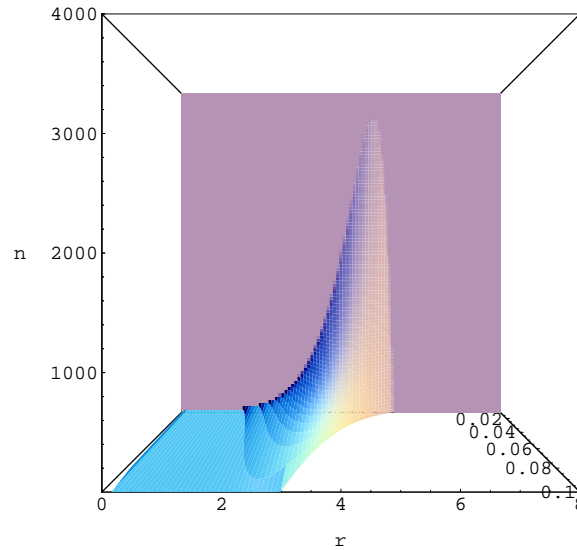
If we're clever and take our Schmidt decomposition basis to be of the form $\sum_k^{n-1} |2k\rangle|2n-2k-1\rangle \langle 2n-2k-1| \langle 2k|$, we can see that exactly n states, one for each $k = 0..n-1$ contribute. Consequently, the Schmidt number of $|\phi_1\rangle = B|\psi\rangle$ is n .

What is the Schmidt number of the state given by feeding a coherent state $|\alpha\rangle$ and vacuum $|0\rangle$ into B , i.e. $|\psi_1\rangle = B|\alpha\rangle|0\rangle$?

Note that our $|\psi_1\rangle = |\alpha/\sqrt{2}\rangle|\alpha/\sqrt{2}\rangle = \sum_{n,m} e^{-|\alpha|^2/2} \left(\frac{\alpha/\sqrt{2}}{\sqrt{n!}} \right)^n \left(\frac{\alpha/\sqrt{2}}{\sqrt{m!}} \right)^m |n\rangle|m\rangle$, so when we write out the density matrix of this state as we did for the Yurke state, we'll find that an infinite number of $|n'\rangle|m'\rangle$ states will have overlap with it. Consequently, the Schmidt number of the state is infinite. However (!) the coefficients of these states become vanishingly small. We'll explore in part e a little more about how the Schmidt number might not be the best measure of entanglement due to that property.

- e) Often, the Schmidt number can lack meaning as a quantitative measure of entanglement, because it includes even the smallest non-zero coefficients in its count. One improvement on this is to count only coefficients above a certain threshold, as illustrated by the following example.

A two-mode squeezed state can be generated in the laboratory by a certain kind of



optical parametric oscillator; suppose this is the state

$$|\Psi\rangle = \exp\left[-\frac{r}{2}(a_1 a_2 - a_1^\dagger a_2^\dagger)\right] |0\rangle_1 |0\rangle_2 \propto \frac{1}{\cosh r} \sum_n \tanh^n r |n\rangle_1 |n\rangle_2. \quad (65)$$

Plot the number of Schmidt coefficients of $|\Psi\rangle$ which are above a fixed threshold, say 0.01, as a function of the squeezing parameter r . Show that this measure of entanglement increases with increasing r , as intuitively desired.

The concept behind this truncation method leads to the truncations used in efficient classical simulation of coupled spin systems, with the density matrix renormalization group and matrix product state techniques.

Here's a plot of the Schmidt coefficients $\frac{\tanh^n(r)}{\cosh(r)}$ with $r(0 \rightarrow 8)$ and $n(0 \rightarrow 4000)$ truncated at 0.01 on the low end. Notice that the number of Schmidt coefficients greater than 0.01 continues to rise with increasing r until one reaches the point that the average Schmidt coefficient falls below 0.01 due to the number of states $|n\rangle_1 |n\rangle_2$ that have sizeable overlap with the heavily-squeezed vacuum.