MIT 8.422 Atomic Physics II — *Spring, 2009*
Prof. Wolfgang Ketterle — Febuary 25, 2009
Prof. Isaac Chuang

Assignment #3

Due: Wednesday, March 11, 2009
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Office Hours: March 2nd (Mon) & March 3rd (Tue), 6pm - 8pm.

1. **One atom and one photon: spontaneous emission**

   A single atom coupled to a single mode of electromagnetic radiation undergoes spontaneous emission. What is the state of the atom during such spontaneous emission?

   Let us model the interaction of one atom with a single optical mode using the Jaynes-Cummings interaction,

$$H = \hbar\omega a^\dagger a + \delta\sigma_z + g(a^\dagger\sigma_- + a\sigma_+)\,, \tag{1}$$

   where δ is the detuning of the cavity from the atom, ω is the cavity frequency, and g is the coupling of the atom to the field. Restricted to the case where at most one quantum is exchanged with the optical mode, we may write this Hamiltonian as a matrix,

$$H = -\begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{bmatrix}\,, \tag{2}$$

   where the basis states are $|0g\rangle$, $|0e\rangle$, and $|1g\rangle$, where $|e\rangle$ and $|g\rangle$ are the ground and excited states of the atom, and the left 0 and 1 labels denote the number of photons in the optical mode.

   a) Compute the full unitary transform for evolution under this Hamiltonian, $U = e^{-iHt}$ and obtain

$$\begin{aligned} U \;=\; & e^{-i\delta t}|0g\rangle\langle 0g| + \left(\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t\right)|0e\rangle\langle 0e| \\ & + \left(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t\right)|1g\rangle\langle 1g| - i\frac{g}{\Omega}\sin\Omega t\left(|0e\rangle\langle 1g| + |1g\rangle\langle 0e|\right)\,, \end{aligned} \tag{3}$$

   where $\Omega = \sqrt{g^2+\delta^2}$ is the Rabi frequency.

   b) Suppose the atom starts out in the excited state $|e\rangle$, and the cavity with no photon, $|0\rangle$. What is the state of the atom after time t, *if the cavity is measured and found to have no photon*? What if one photon is found to be in the cavity?

   c) A reduced density matrix describes the state of part of a system, averaged over the possible states of the remainder. Give a reduced density matrix describing the state of the atom at time t.

   d) Let $|e\rangle$ and $|g\rangle$ be depicted as the south and north poles of a Bloch sphere representation of the atom. Plot the points $(|e\rangle + |g\rangle)/\sqrt{2}$, $(|e\rangle - |g\rangle)/\sqrt{2}$, and $|e\rangle$. Suppose that the atom interacts with the cavity for a short time t (and the cavity starts in $|0\rangle$), after which the cavity is measured. Recall that a two-dimensional density matrix ρ can be represented by a point $\vec{r}$ *inside* the Bloch sphere, using

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}\,. \tag{4}$$

Plot how these three initial states evolve under repeated short evolutions with the cavity. The cavity state is reset to $|0\rangle$ after each interaction. What is the fixed point of this process?

2. **Driven two-level atom: dressed states**

A two-level atom driven by a classical laser field is often conveniently studied in a stationary basis, that is, the basis defined by the eigenstates of the Hamiltonian. These basis states are known as *dressed states*, and they are useful for interpreting many phenomena and solving problems in atomic physics.

Let the Hamiltonian for the classically driven atom be

$$H = \frac{\hbar\omega_0}{2}Z + \frac{\hbar\Omega_1}{2}\Big[X\cos(\omega_L t) + Y\sin(\omega_L t)\Big]\,, \tag{5}$$

where X, Y, and Z are the usual Pauli matrices, $\hbar\omega_0$ is the energy difference between the atomic levels $|e\rangle$ and $|g\rangle$, ω_L is the laser frequency, and $\Omega_1 = eE_0\langle g|z|e\rangle/\hbar$ is the Rabi frequency.

a) Write the coupled time-dependent Schrodinger equations using a solution anzatz of the form $|\psi(t)\rangle = ae^{i\omega_1 t}|g\rangle + be^{i\omega_2 t}|e\rangle$. Choose the frequencies such that the equations are steady-state, containing no oscillating terms.

b) The Schrodinger equations you have just written are identical to those for a system with a Hamiltonian that is a function of the Rabi frequency and the detuning $\delta_L = \omega_L - \omega_0$ only. Give this Hamiltonian; denote it as H'.

c) Write H' in terms of trigonometric functions, where $\sin 2\theta = \Omega_1/\Omega$, where $\Omega = \sqrt{\Omega_1^2 + \delta_L^2}$ is the "effective" Rabi frequency.

d) Diagonalize H' to find the eigenvalues and associated eigenvectors.

e) Use these results to find the time-dependent solutions to the Schrodinger equations for H, in the original frame of reference.

3. **Master equation for a damped optical cavity**

A Fabry-Perot cavity can be modeled as being made of a high reflectivity mirror and a perfect mirror with fixed spacing. Clearly, photons stored inside this cavity will gradually leak out the partially reflecting mirror, causing the state inside to change. This process is described by a master equation, much like an atom coupled to fields is described by the optical Bloch equation. In this problem, we explore a simple derivation of such a master equation, for a single mode cavity.

a) Let a and $a^\dagger$ describe the optical mode of interest within the cavity, with characteristic energy $\hbar\omega$. Let $|\psi\rangle$ describe the initial cavity state.

Let us suppose that photons leak out of the cavity at a rate proportional to the photon number in the cavity, and to Γ which parameterizes the leakiness of the leaky mirror.

Thus, for short times dt, the probability of a photon leaking from the cavity is $dp = \Gamma dt \langle\psi|a^\dagger a|\psi\rangle$.

Another way to write this is to say that if a photon leaks over a time interval dt, the cavity state changes to become the un-normalized state $a\sqrt{\Gamma\, dt}|\psi\rangle$, while if no photon leaks during time dt, we can model the state as having evolved under the "Hamiltonian" $H = H_0 - i(\hbar\Gamma/2)a^\dagger a$ (where $H_0 = \hbar\omega a^\dagger a$ is the usual cavity evolution Hamiltonian).

Note that H is not Hermitian. It has an imaginary term, so it is "lossy". Evolving a state forward in time with H yields an un-normalized state.

The two un-normalized states will later become two "components" of the density matrix. Give $|\psi_0\rangle$, the normalized state obtained if no photon leaks, and $|\psi_1\rangle$, the normalized state obtained if one photon leaks. Provide some words of justification for the lossy term in H (for instance, why is it proportional to $a^\dagger a$?).

b) At time t, suppose the cavity starts in the pure quantum state $|\psi\rangle$. At time $t + dt$, for short times dt, the density matrix of the leaky cavity system is given by

$$\rho(t + dt) = (1 - dp)|\psi_0\rangle\langle\psi_0| + dp|\psi_1\rangle\langle\psi_1|\,. \tag{6}$$

Rewrite this density matrix, building it out of the the un-normalized states described in part a). The only bras or kets to appear in your final expression should be $|\psi\rangle$ and $\langle\psi|$. Where you encounter exponentials of H or $H^\dagger$ (which is NOT equal to H !), expand them to first order in dt.

c) Compute $\rho(t + dt) - \rho(t)$ for small dt, and write the coarse-grained differential equation

$$\frac{d}{dt}\rho(t) \approx \frac{\rho(t + dt) - \rho(t)}{dt}\,. \tag{7}$$

Show that this has the proper Lindblad form of

$$\frac{d\rho}{dt} = \frac{i}{\hbar}[\rho, H_0] - C^\dagger C\rho - \rho C^\dagger C + 2C\rho C^\dagger\,, \tag{8}$$

and identify what C and $C^\dagger$ are for this case.