MIT 8.422 Atomic Physics II *Spring, 2009*
Prof. Wolfgang Ketterle Febuary 25, 2009
Prof. Isaac Chuang

Assignment #3 Solutions

Due: Wednesday, February 25, 2009
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1. **One atom and one photon: spontaneous emission**

   A general remark for those who were wondering how to derive equation (4) from equation (3): When you calculate the matrix from the Hamiltonian (3) instead of the given matrix you will obtain (order $|0g\rangle, |1g\rangle, |0e\rangle$):

$$\begin{pmatrix} \delta & 0 & 0 \\ 0 & \hbar\omega + \delta & g \\ 0 & g & -\delta \end{pmatrix}$$
$$= - \begin{pmatrix} -\delta & 0 & 0 \\ 0 & -\hbar\omega - \delta & -g \\ 0 & -g & \delta \end{pmatrix}$$

   You can get rid of the $\hbar\omega$ by transforming to the rotating frame. If you then redefine $\delta \to -\delta$ and $g \to -g$, you obtain equation (4).

   (a) (2 points) Compute the full unitary transform for evolution under the given Hamiltonian:
   Solution:

$$\begin{aligned} U &= e^{-iHt} \\ &= \exp(i \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{pmatrix} t) \\ &= 1 + it \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{pmatrix} + \frac{(it)^2}{2} \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{pmatrix}^2 + ... \\ &\boxed{= \begin{pmatrix} e^{i\delta t} & 0 & 0 \\ 0 & \cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t & i\frac{g}{\Omega}\sin\Omega t \\ 0 & i\frac{g}{\Omega}\sin\Omega t & \cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t \end{pmatrix}} \end{aligned}$$

   which is the desired expression and where $\Omega = \sqrt{\delta^2 + g^2}$. The last equality follows from inspection or Mathematica's "MatrixExp"-function.

   (b) (2 points) Suppose the atom starts out in the excited state $|e\rangle$, and the cavity with no photon, $|0\rangle$. What is the state of the atom after time t, if the cavity is measured and found to have no photon? What if one photon is found to be in the cavity?

Solution:

$$
\begin{aligned}
U = e^{i\delta t}|0g\rangle\langle 0g| &+ (\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t)|0e\rangle\langle 0e| \\
&+ (\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t)|1g\rangle\langle 1g| + i\frac{g}{\Omega}\sin\Omega t(|0e\rangle\langle 1g| + |1g\rangle\langle 0e|)
\end{aligned}
$$

The atom starts in state $|0e\rangle$ and at the time of the measurement t_0 the state of the cavity is measured to be $|0\rangle$. From U it can quickly be seen that the atom must be in the excited state $|e\rangle$.
The probability for this event is given by:

$$
\begin{aligned}
t_0 : P_e = |\langle 0|U|0e\rangle|^2 &= \left|\cos\Omega t_0 + i\frac{\delta}{\Omega}\sin\Omega t_0|e\rangle\right|^2 \\
&= \cos^2\Omega t_0 + \frac{\delta^2}{\Omega^2}\sin^2\Omega t_0 \qquad \text{...in state } |e\rangle
\end{aligned}
$$

Similarly, if one photon is found in the cavity the atom must be in state $|g\rangle$ with the probability:

$$
\begin{aligned}
t_0 : P_g = |\langle 1|U|0e\rangle|^2 &= \left|i\frac{g}{\Omega}\sin\Omega t_0|g\rangle\right|^2 \\
&= \frac{g^2}{\Omega^2}\sin^2\Omega t_0 \qquad \text{...in state } |g\rangle
\end{aligned}
$$

Check normalization: $P_g + P_e = \cos^2\Omega t_0 + \frac{\delta^2+g^2}{\Omega^2}\sin^2\Omega t_0 = 1$

(c) (4 points) Give a reduced density matrix describing the state of the atom at time t.

Solution:

The time evolution of the density matrix $\rho(t)$ is given by: $\rho(t) = U\rho_0 U^\dagger$. We are interested in the reduced density matrix $\rho_{red}(t) = \langle 0|\rho_0|0\rangle + \langle 1|\rho_0|1\rangle$.
Generally, the density matrix at t=0 can be written as:

$$
\begin{aligned}
\rho_0 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes |0\rangle\langle 0| \\
&= a|0g\rangle\langle 0g| + b|0g\rangle\langle 0e| + c|0e\rangle\langle 0g| + d|0e\rangle\langle 0e|
\end{aligned}
$$

Putting everything together, we have:

$$
\begin{aligned}
\langle 0|U\rho_0 U^\dagger|0\rangle = a|g\rangle\langle g| &+ be^{i\delta t}(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t)|g\rangle\langle e| \\
&+ ce^{-i\delta t}(\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t)|e\rangle\langle g| \\
&+ d(\cos^2\Omega t + \frac{\delta^2}{\Omega^2}\sin^2\Omega t)|e\rangle\langle e|
\end{aligned}
$$

and:

$$\langle 1|U\rho_0 U^\dagger|1\rangle = d\frac{g^2}{\Omega^2}\sin^2\Omega t \;\; |e\rangle\langle e|$$

Or, in matrix form:

$$\boxed{\rho_{red}(t) = \begin{pmatrix} a + d\,\frac{g^2}{\Omega^2}\sin^2\Omega t & b\,e^{i\delta t}(\cos\Omega t - i\,\frac{\delta}{\Omega}\sin\Omega t) \\ c\,e^{-i\delta t}(\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t) & d\,(\cos^2\Omega t + \frac{\delta^2}{\Omega^2}\sin^2\Omega t) \end{pmatrix}} \qquad (1)$$

Check normalization: $\mathrm{Tr}(\rho_{red}(t)) = a + d = 1$

(d) (3 points) Plot how the three given initial states evolve under repeated short evolutions with the cavity. What is the fixed point of this process?

Solution:

For short times ($\delta\Delta t \ll 1$ and $g\Delta t \ll 1$), sine and cosine can be approximated. However, care has to be taken in order to maintain $\mathrm{Tr}(\rho_{red}(t)) = 1$! Let's try:

$$\begin{aligned} \sin\Omega\Delta t &\simeq \Omega\Delta t \\ \sin^2\Omega\Delta t &\simeq \Omega^2\Delta t^2 \\ \cos\Omega\Delta t &\simeq 1 - \frac{\Omega^2\Delta t^2}{2} \\ \cos^2\Omega\Delta t &\simeq 1 - \Omega^2\Delta t^2 \\ e^{\pm i\,\delta\Delta t} &\simeq 1 \pm i\,\delta\Delta t - \frac{\delta^2\Delta t^2}{2} \\ \Rightarrow e^{\pm i\,\delta\Delta t}(\cos\Omega\Delta t \mp i\,\frac{\delta}{\Omega}\sin\Omega\Delta t) &= 1 - \frac{g^2\Delta t^2}{2} + O(\Delta t^3) \end{aligned}$$

(Remark: The factor of 2 difference between the expansion of $\cos x$ vs $\cos^2 x$ will turn out to be responsible for the different damping rates of coherences and populations)
So for short times, the entries of the density matrix can be approximated as:

$$\rho_{red}(t + \Delta t) = \begin{pmatrix} a + d\,g^2\Delta t^2 & b\,(1 - \frac{g^2}{2}\Delta t^2) \\ c\,(1 - \frac{g^2}{2}\Delta t^2) & d\,(1 - g^2\Delta t^2) \end{pmatrix}$$

Note that normalization is indeed conserved for the chosen approximation.
We can now calculate how the density matrix evolves as a function of time if after short times of free evolution the cavity is reset to $|0\rangle$ using the mapping $a(t+\Delta t) = a(t)\,g^2\Delta t^2$, $b(t+\Delta t) = b(t)\,(1-\frac{g^2}{2}\Delta t^2)$, $c(t+\Delta t) = c(t)\,(1-\frac{g^2}{2}\Delta t^2)$ and $d(t+\Delta t) = d(t)\,(1-g^2\Delta t^2)$.

But for simplicity, let's switch to the Bloch sphere picture:

$$\begin{aligned}
r_x(\Delta t) &= \rho_{eg} + \rho_{ge} = \left(1 - \frac{g^2 \Delta t^2}{2}\right)\,(b+c) = \left(1 - \frac{g^2 \Delta t^2}{2}\right)\, r_x(t=0) \\
r_y(\Delta t) &= \frac{\rho_{eg} - \rho_{ge}}{i} = \left(1 - \frac{g^2 \Delta t^2}{2}\right) \frac{1}{i}(b-c) = \left(1 - \frac{g^2 \Delta t^2}{2}\right)\, r_y(t=0) \\
r_z(\Delta t) &= \rho_{ee} - \rho_{gg} = (a-d) + d\, g^2 \Delta t^2 = g^2 \Delta t^2 + (1 - g^2 \Delta t^2)\, r_z(t=0)
\end{aligned}$$

Or, after N time steps:

$$\begin{aligned}
r_x(N\,\Delta t) &= \left(1 - \frac{g^2 \Delta t^2}{2}\right)^N r_x(0) \\
&= r_x(0)\, e^{N\,\ln(1 - \frac{g^2 \Delta t^2}{2})} \\
&\simeq r_x(0)\, e^{-\frac{g^2 \Delta t}{2}\, N \Delta t} \\
&\boxed{= r_x(0)\, e^{-\frac{\Gamma}{2}\, t}}
\end{aligned}$$

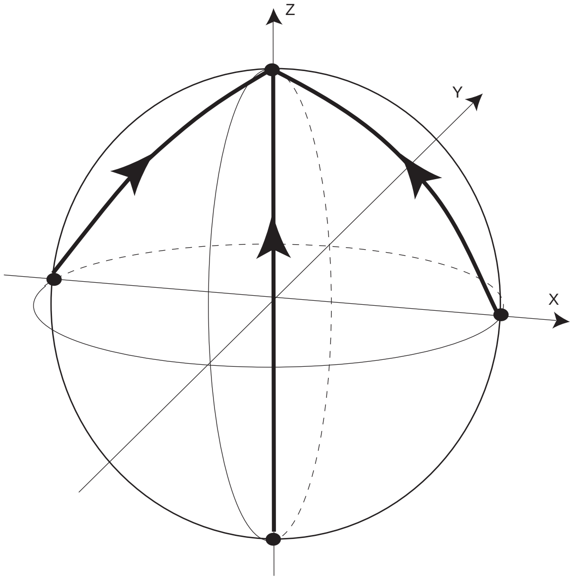
where $t := N\,\Delta t$ and $\Gamma := g^2 \Delta t$. Similarly, the time dependence of $r_y(t)$ can be calculated. For $r_z(t)$ it can be shown that:

$$\begin{aligned}
r_z(N\,\Delta t) &= g^2 \Delta t^2 \sum_{k=0}^{N-1} (1 - g^2 \Delta t^2)^k + (1 - g^2 \Delta t^2)^N\, r_z(0) \\
&= g^2 \Delta t^2 \frac{1 - (1 - g^2 \Delta t^2)^N}{1 - (1 - g^2 \Delta t^2)} + (1 - g^2 \Delta t^2)^N r_z(0) \\
&= 1 - r_z(0)(1 - g^2 \Delta t^2)^N \\
&\boxed{= 1 - r_z(0)\, e^{-\Gamma\, t}}
\end{aligned}$$

As discussed in class, we see that the populations decay twice as fast as the coherences. The three cases in the problem set differ only in their initial conditions. Time Evolution: See figure.

i.

$$\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|e\rangle \\
\Rightarrow \rho_0 &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} r_x^{(0)} \\ r_y^{(0)} \\ r_z^{(0)} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$



ii.

$$
\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}|g\rangle - \frac{1}{\sqrt{2}}|e\rangle \\
\Rightarrow \rho_0 &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} r_x^{(0)} \\ r_y^{(0)} \\ r_z^{(0)} \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

iii.

$$
\begin{aligned}
|\psi_0\rangle &= |e\rangle \\
\Rightarrow \rho_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} r_x^{(0)} \\ r_y^{(0)} \\ r_z^{(0)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}
\end{aligned}
$$

2. **Driven two-level atom: dressed states**

   a) Write the coupled time-dependent Schrodinger equations using a solution anzatz of the form $|\psi(t)\rangle = ae^{i\omega_1 t}|g\rangle + be^{i\omega_2 t}|e\rangle$.

Solution:

If we apply the Hamiltonian H to the wavefunction $\Psi(t)$ in the time-dependent Schrodinger equation $i\hbar\frac{\partial\Psi(t)}{\partial t} = H\Psi(t)$, we obtain the 2 equations:

$$i\dot{a} + a(-\omega_1 + \omega_0/2) = \frac{\Omega_1}{2} b \exp(i(\omega_2 + \omega_L - \omega_1)t)$$

and

$$i\dot{b} + b(-\omega_2 - \omega_0/2) = \frac{\Omega_1}{2} a \exp(i(\omega_1 - \omega_L - \omega_2)t).$$

The oscillating terms can be eliminated by selecting $\omega_1 = \omega_L/2$ and $\omega_2 = -\omega_L/2$. The equations can then be written

$$i\dot{a} = a\frac{\delta_L}{2} + b\frac{\Omega_1}{2}$$

$$i\dot{b} = -b\frac{\delta_L}{2} + a\frac{\Omega_1}{2},$$

where $\omega_L = \omega_0 + \delta_L$.

b) The Hamiltonian corresponding to the two preceeding equations for a and b is:

$$H' = \frac{\hbar}{2}\begin{pmatrix} \delta_L & \Omega_1 \\ \Omega_1 & -\delta_L \end{pmatrix}$$

c) Using appropriate trig identities, the new Hamiltonian becomes:

$$H = \frac{\hbar\Omega}{2}\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

d) Using a little sweat or Mathematica, the eigenvalues of the above matrix $E_\pm = \pm\frac{\hbar\Omega}{2}$, and the eigenvectors are $|\psi_-\rangle = \binom{-\sin(\theta)}{\cos(\theta)}$ and $|\psi_+\rangle = \binom{\cos(\theta)}{\sin(\theta)}$.

e)

$$|\psi(t)\rangle = c_+|\psi_+\rangle \exp(-i\Omega t) + c_-|\psi_-\rangle \exp(i\Omega t) =$$

$$(\cos(\theta)\exp(it\omega_L/2)|g\rangle + \sin(\theta)\exp(-it\omega_L/2))\exp(-i\Omega t) +$$

$$(-\sin(\theta)\exp(it\omega_L/2)|g\rangle + \cos(\theta)\exp(-it\omega_L/2)\exp(i\Omega t)$$

3. **Master equation for a damped optical cavity**

a) Let's define the "un-normalized state with a photon leak" as $|\tilde{\psi_1}\rangle = \sqrt{\Gamma dt} a|\psi\rangle$, and the "un-normalized state without a photon leak" as $|\tilde{\psi_0}\rangle = \exp(-iHdt/\hbar)|\psi\rangle$.

   To normalize see that $\langle\tilde{\psi_1}|\tilde{\psi_1}\rangle = \Gamma dt\langle\psi|a^\dagger a|\psi\rangle$, thus $|\psi_1\rangle = \frac{1}{\sqrt{\Gamma dt} < a^\dagger a >} a|\psi\rangle$.

   Meanwhile, $\langle\tilde{\psi_0}|\tilde{\psi_0}\rangle = \langle\psi|\exp(iH^\dagger dt/\hbar)\exp(-iHdt/\hbar)|\psi\rangle = \langle\psi|exp(-2a^\dagger a dt/\hbar)|\psi\rangle$ (to first order in dt, at least). So $|\psi_0\rangle = \frac{\exp(-iHdt/\hbar)|\psi\rangle}{\langle exp(-2a^\dagger a dt/\hbar\rangle}$.

b) Starting from the un-normalized states defined in part a), we have

   $\rho(t+dt) = |\tilde{\psi_0}\rangle\langle\tilde{\psi_1}| + |\tilde{\psi_0}\rangle\langle\tilde{\psi_0}| = \Gamma dt a|\psi\rangle\langle\psi|a^\dagger + \exp(-iHdt/\hbar)|\psi\rangle\langle\psi|\exp(iH^\dagger dt/\hbar)$

   Expanding to first order in dt, we have

   $\Gamma dt a|\psi\rangle\langle\psi|a^\dagger + (1 - iH_0 dt/\hbar - \frac{\Gamma dt}{2} a^\dagger a)|\psi\rangle\langle\psi|(1 + iH_0 dt/\hbar - \frac{\Gamma}{2} a^\dagger a)$

   Simplifying this expression slightly, and again dropping terms of higher order than dt, we get

   $\rho(t+dt) = \Gamma dt a|\psi\rangle\langle\psi|a^\dagger + |\psi\rangle\langle\psi| - \frac{idt}{\hbar}[H_0, |\psi\rangle\langle\psi|] - \frac{\Gamma dt}{2}(a^\dagger a|\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|a^\dagger a)$

c) Taking the coarse-grained derivative of the above expression, we get

   $\frac{d\rho}{dt} = \frac{i}{\hbar}[\rho, H_0] + \Gamma a\rho a^\dagger - \frac{\Gamma}{2}(a^\dagger a\rho + \rho a a^\dagger)$

   This expression has the desired Lindblad form, for $C = a\sqrt{\Gamma/2}$, $C^\dagger = a^\dagger\sqrt{\Gamma/2}$.