*MIT 8.422 Atomic Physics II* — *Spring, 2009*
Prof. Wolfgang Ketterle — March 4, 2009
Prof. Isaac Chuang

# Assignment #4

Due: Wednesday, March 11, 2009
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Office Hours: March 9th (Mon) & March 10th (Tue), 6pm - 8pm.

1. **Physical origin of $T_2$.**

   What is the physical meaning of the decay of the off-diagonal elements of the atomic density matrix, $\rho_{eg}$ and $\rho_{ge}$, due to spontaneous emission and other processes? Three very simple models for this kind of quantum noise, which leads to loss of *phase coherence*, and gives rise to decoherence times characterized by $T_2$, are the following.

   a) One physical origin for the decay of the off-diagonal elements of the atomic density matrix (this decay is known as "phase damping") is random phase noise. Suppose that we have an atom in the arbitrary state

   $$\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{1}$$

   excited by far off-resonance light of random intensity. The effect of this light on the state is an AC stark shift $|e\rangle \to e^{i\theta}|e\rangle$, which we may model as a rotation $R_z(\theta)$, where the angle of rotation $\theta$ is random, distributed as a Gaussian with mean 0 and variance $2\lambda t$. Give $\rho(\theta) = R_z(\theta)\rho R_z^\dagger(\theta)$, and the expected density matrix $\bar{\rho}(t) = \langle\rho(\theta)\rangle$, averaged over this Gaussian distribution.

   b) Another physical origin for phase damping is elastic collisions. Assume the two-level atom bounces along a waveguide, interacting with the walls without loosing kinetic energy, but changing its trajectory slightly at each bounce, in a manner depending on the state of the atom. This can be modeled by a Hamiltonian interacting the atom with a single mode environment,

   $$H_{SE} = |e\rangle\langle e| \otimes \Big[\gamma|0\rangle\langle 1| + \gamma^*|1\rangle\langle 0|\Big], \tag{2}$$

   with coupling constant $\gamma$. Compute the evolution of an initial atomic state $a|g\rangle + b|e\rangle$ coupled to an environment $|0\rangle$, evolved for a small differential timestep $dt$, and give the density matrix of the atom $\rho(t)$ for small $t$.

   c) A third physical model for phase damping is the following scenario (which you may think a bit unphysical). Suppose the two-level atom is subject to a force which randomly flips the phase of the atom by $-1$, changing $|e\rangle \to -|e\rangle$, with probability $(1-e^{-\lambda t})/2$ at each moment in time. Assuming the density matrix is initially arbitrary, give $\rho(t)$ averaged over this random phase flip process.

   An amazing fact about these three models is that if phase damping is known to be happening to an atom, say during traversal through some black box, without additional knowledge about what is inside the black box, there is *in principle, no way to distinguish* between which of the above physical processes is causing the phase damping! They are perfectly equivalent and equally legitimate. This is fundamental to why quantum error correction can work: phase noise can be modeled as either no error happens to the state, or a phase flip "error" occurs (analogous to bit errors in a communication channel), even if the actual physical process is a different one.

2. **Optical Bloch Equations: weak and short-time limits**.

The time-independent form of the optical Bloch equations (eg, see API p. 359), including spontaneous emission, and the rotating wave approximation, is:

$$\dot{\rho}_{ee} = i\frac{\Omega}{2}(\rho_{eg} - \rho_{ge}) - \Gamma\rho_{ee} \tag{3}$$

$$\dot{\rho}_{ge} = i(\omega_0 - \omega_L)\rho_{ge} - i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg}) - \frac{\Gamma}{2}\rho_{ge}\,, \tag{4}$$

where the remaining two components of the density matrix are given by $\rho_{gg} = 1 - \rho_{ee}$, and $\rho_{eg} = \rho_{ge}^*$. It is insightful to study these equations in the limit of weak excitation, and for short evolution times.

a) Show that the solution of these equations to lowest order in $|\Omega|$, and in the limit $|\Omega| \ll \Gamma$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{\frac{1}{4}|\Omega|^2}{(\omega_0 - \omega_L)^2 + \left(\frac{\Gamma}{2}\right)^2}\left[1 + e^{-\Gamma t} - 2\cos[(\omega_0 - \omega_L)t]e^{-\Gamma t/2}\right]\,. \tag{5}$$

What does this solution reduce to in the limit of an infinitely narrow linewidth ($\Gamma \to 0$)?

b) Show that the solution of these equations to lowest order in $|\Omega|$ in the limit $|\Omega|t \ll 1$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{1}{4}|\Omega|^2 t^2\,, \tag{6}$$

irrespective of the values of $(\omega_0 - \omega_L)$ and $\Gamma$. Why?

3. **Single photon generation from resonance fluorescence of a single atom**

A two-level atom can emit only one photon at a time. Consider a atom driven by a classical field with Rabi frequency $g$ which is small compared with its spontaneous emission rate $\Gamma$. After a spontaneous emission event, the atom is initially in the ground state, so on average it takes time on the order of $\sim 1/g$ for the atom to Rabi flop back into its excited state, so that it can emit once again. The photon statistics of the atom's fluorescence thus show a strong anti-bunching effect, as you see in this problem via analysis of the $g_2(\tau)$ function for resonance fluorescence.

Recall the definition of $g^{(2)}(\tau)$ for a single optical mode:

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger(t)a(t)\rangle^2}\,. \tag{7}$$

If the optical mode is generated from a single two-level atom, this may be expressed as

$$g^{(2)}(\tau) = g_0^{(2)}\langle\sigma_+(t)\sigma_+(t+\tau)\sigma_-(t+\tau)\sigma_-(t)\rangle\,, \tag{8}$$

where $g_0^{(2)}$ is constant. (You can get some insight why this is so by considering how to write the $a$ and $a^\dagger$ operators in the Heisenberg picture)

(a) The *quantum regression theorem* (also known as the Onsager-Lax regression theorem) states that for operators $A(t)$, $B(t)$, and $C(t)$, if

$$\langle A(t)\rangle = \sum_k \alpha_k(t)\langle A_k(0)\rangle \,, \tag{9}$$

then

$$\langle B(0)A(t)C(0)\rangle = \sum_k \alpha_k(t)\langle B(0)A_k(0)C(0)\rangle \,. \tag{10}$$

This powerful theorem allows two-time correlation functions to be determined by density matrix solutions to the master equation. Assume that we have a solution $\rho(t)$ to the master equation describing the driven atom. Suppose that

$$\rho_{ee}(t) = \alpha_1(t) + \alpha_2(t)\langle\rho_{ge}(0)\rangle + \alpha_3(t)\langle\rho_{eg}(0)\rangle + \alpha_4(t)\langle\rho_{ee}(0)\rangle \,, \tag{11}$$

where $\lim_{t\to\infty}\alpha_k(t) = 0$ for all but $k = 1$. Use the quantum regression theorem to show that in this case,

$$g^{(2)}(\tau) = \frac{\langle\sigma_+(0)\sigma_+(\tau)\sigma_-(\tau)\sigma_-(0)\rangle}{[\langle\sigma_+(\infty)\sigma_-(\infty)\rangle]^2} = \frac{\alpha_1(\tau)}{\alpha_1(\infty)} \,. \tag{12}$$

(b) One approximate solution to the master equation for this scenario is given by dropping all but the spontaneous emission decay terms for the excited state population, namely using the rate equation

$$\dot\rho_{ee} \approx g - \Gamma\rho_{ee} \,. \tag{13}$$

Use the solution of this differential equation to compute an approximation to $g^{(2)}(\tau)$.

(c) Use the full solution to the optical Bloch equation, in the weak excitation limit (problem 2, above), to compute $g^{(2)}(\tau)$. Plot this for various interesting values of $\Gamma/\Omega$, at fixed detuning ($\omega_0 - \omega_L = \delta$).

(d) In a few sentences, explain what the value of $g^{(2)}(\tau)$ has to do with "photon anti-bunching". In this context, compare the behavior of $g^{(2)}(\tau)$ for the situations calculated above and for a classical light field.