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Assignment #4 Solution

Due: Wednesday, March 11, 2009
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1. **Physical origin of T_2.**

   a) One physical origin for the decay of the off-diagonal elements of the atomic density matrix (this decay is known as "phase damping") is random phase noise. Suppose that we have an atom in the arbitrary state

$$\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{1}$$

excited by far off-resonance light of random intensity. The effect of this light on the state is an AC stark shift $|e\rangle \to e^{i\theta}|e\rangle$, which we may model as a rotation $R_z(\theta)$, where the angle of rotation θ is random, distributed as a Gaussian with mean 0 and variance $2\lambda t$. Give $\rho(\theta) = R_z(\theta)\rho R_z^\dagger(\theta)$, and the expected density matrix $\bar{\rho}(t) = \langle \rho(\theta) \rangle$, averaged over this Gaussian distribution.

Let's start by writing out the rotation matrix product:

$$\rho(\theta) = \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} = \begin{bmatrix} a & be^{i\theta} \\ ce^{-i\theta} & d \end{bmatrix}$$

We now average this over the probability distribution

$$P(\theta) = \frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{\theta^2}{4\lambda t}}$$

by completing the square in the exponent:

$$\langle e^{\pm i\theta} \rangle = \int \frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{\theta^2}{4\lambda t}} e^{\pm i\theta} d\theta = e^{-\lambda t} \int \frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{(\theta \mp 2i\lambda t)^2}{4\lambda t}} d\theta = e^{-\lambda t}.$$

Consequently,

$$\bar{\rho}(t) = \langle \rho(\theta) \rangle = \begin{bmatrix} a & be^{-\lambda t} \\ ce^{-\lambda t} & d \end{bmatrix}.$$

b) Another physical origin for phase damping is elastic collisions. Assume the two-level atom bounces along a waveguide, interacting with the walls without loosing kinetic energy, but changing its trajectory slightly at each bounce, in a manner depending on the state of the atom. This can be modeled by a Hamiltonian interacting the atom with a single mode environment,

$$H_{SE} = |e\rangle\langle e| \otimes \Big[\gamma|0\rangle\langle 1| + \gamma^*|1\rangle\langle 0|\Big] \,, \tag{2}$$

with coupling constant γ. Compute the evolution of an initial atomic state $a|g\rangle + b|e\rangle$ coupled to an environment $|0\rangle$, evolved for a small differential timestep dt, and give the density matrix of the atom $\rho(t)$ for small t.

We know that this Hamiltonian mixes only the $|e0\rangle$ and $|e1\rangle$ states, so the stationary states of the system are:

$$|g0\rangle \ , |g1\rangle \ , \frac{e^{i\frac{\phi}{2}}\,|e0\rangle + e^{-i\frac{\phi}{2}}\,|e1\rangle}{\sqrt{2}} \ , \frac{e^{i\frac{\phi}{2}}\,|e0\rangle - e^{-i\frac{\phi}{2}}\,|e1\rangle}{\sqrt{2}}$$

with energies $0\,,0\,,|\gamma|\,,-\gamma$, where we let $\gamma \rightarrow |\gamma|e^{i\phi}$. This allows us to write out the propagator

$$U = |g0\rangle\langle g0| + |g1\rangle\langle g1| + \cos(|\gamma|t)[|e0\rangle\langle e0| + |e1\rangle\langle e1|] - i\sin(|\gamma|t)[e^{i\phi}\,|e0\rangle\langle e1| + e^{-i\phi}\,|e1\rangle\langle e0|]\,.$$

Applying this to the given initial state $|\Psi\rangle = a\,|g0\rangle + b\,|e0\rangle$ results in

$$|\Psi(t)\rangle \ = \ a\,|g0\rangle + b\cos(|\gamma|t)\,|e0\rangle - ib\sin(|\gamma|t)e^{-i\phi}\,|e1\rangle \ .$$

Tracing this over the environment:

$$\begin{aligned}\rho(t) \ &= \ Tr_{env}[|\Psi(t)\rangle\langle\Psi(t)|] = \langle 0|\,|\Psi(t)\rangle\langle\Psi(t)|\,|0\rangle + \langle 1|\,|\Psi(t)\rangle\langle\Psi(t)|\,|1\rangle \\ &= \ |a|^2\,|g\rangle\langle g| + |b|^2\,|e\rangle\langle e| + \cos(|\gamma|t)[ab^*\,|g\rangle\langle e| + ba^*\,|e\rangle\langle g|]\,.\end{aligned}$$

Finally, we approximate this for a short time, δt, and write out a short time approximation

$$\rho(\delta t) \ \approx \ \begin{bmatrix} |a|^2 & (1 - \frac{|\gamma|^2}{2}(\delta t)^2)ab^* \\ (1 - \frac{|\gamma|^2}{2}(\delta t)^2)ba^* & |b|^2 \end{bmatrix}$$

Note that at this point, we're already seeing the characteristic phase decay and no effect on the population. Finally, if we'd like to get back to an exponential damping of the phase terms, we have to step back for a moment and note that we've essentially said there is some small probability in any given time that the off diagonal values will decrease. If we treat that as just a probability for phase damping in any small timestep (with malice of forethought, let's call it λ), and then apply that small evolution N times, we have an exponential generating function

$$\rho(t) \approx \begin{bmatrix} |a|^2 & (1-\frac{\lambda t}{N})^N ab^* \\ (1-\frac{\lambda t}{N})^N ba^* & |b|^2 \end{bmatrix} \rightarrow \begin{bmatrix} |a|^2 & e^{-\lambda t} ab^* \\ e^{-\lambda t} ba^* & |b|^2 \end{bmatrix}$$

c) A third physical model for phase damping is the following scenario (which you may think a bit unphysical). Suppose the two-level atom is subject to a force which randomly flips the phase of the atom by -1, changing $|e\rangle \rightarrow -|e\rangle$, with probability $(1-e^{-\lambda t})/2$ at each moment in time. Assuming the density matrix is initially arbitrary, give $\rho(t)$ averaged over this random phase flip process.

In this case, we start by noting the effect of the $-|e\rangle \rightarrow |e\rangle$ operation on an arbitrary density matrix is

$$\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \rho' = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

As we're given the probability of having flipped an odd number of times $P(t) = \frac{1-e^{-\lambda t}}{2}$, we have all we need to write out the full solution

$$\rho(t) = P(t)\rho' + (1-P(t))\rho = \frac{1-e^{-\lambda t}}{2} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} + (1 - \frac{1-e^{-\lambda t}}{2}) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & be^{-\lambda t} \\ ce^{-\lambda t} & d \end{bmatrix}.$$

2. **Optical Bloch Equations: weak and short-time limits**.

a) Show that the solution of these equations to lowest order in $|\Omega|$, and in the limit $|\Omega| \ll \Gamma$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{\frac{1}{4}|\Omega|^2}{(\omega_0 - \omega_L)^2 + \left(\frac{\Gamma}{2}\right)^2} \left[1 + e^{-\Gamma t} - 2\cos[(\omega_0 - \omega_L)t]e^{-\Gamma t/2}\right]. \quad (3)$$

What does this solution reduce to in the limit of an infinitely narrow linewidth ($\Gamma \rightarrow 0$)?

We begin by noting that as the system starts in the ground state and the pumping is weak, we expect that ρ_{ee} will remain small. Consequently, eqn. (3) can be approximated as

$$\dot{\rho}_{ge} + (\frac{\Gamma}{2} - i(\omega_0 - \omega_L))\rho_{ge} = i\frac{\Omega}{2}\rho_{gg}$$

Now we note that this differential equation can be solved with the method of characteristic multipliers

$$\frac{dy}{dt} + P(t)y = Q(t) \;\Rightarrow\; ye^{\int^t P(t')dt'} = \int^t Q(t'')e^{\int^{t''} P(t')dt'} dt'' + C$$

Which, in this case, is equivalent to noting that

$$\frac{d}{dt}(e^{\alpha t}\rho_{ge}) \;=\; i\frac{\Omega}{2}e^{\alpha t}$$

where $\alpha = \frac{\Gamma}{2} - i(\omega_0 - \omega_l)$. Solving for the given initial condition $\rho_{ge}(0) = 0$:

$$\rho_{ge}(t) = \frac{i\Omega/2}{-i(\omega_0 - \omega_L) + \Gamma/2}(1 - e^{i(\omega_0 - \omega_L - \Gamma/2)t})$$

Plugging this solution for ρ_{ge} into the differential equation for ρ_{ee}, using the method of characteristic multipliers again, and solving for the initial condition $\rho_{ee}(0) = 0$ results in

$$\rho_{ee} = \frac{\frac{1}{4}|\Omega|^2}{(\omega_0 - \omega_L)^2 + \left(\frac{\Gamma}{2}\right)^2}\left[1 + e^{-\Gamma t} - 2\cos[(\omega_0 - \omega_L)t]e^{-\Gamma t/2}\right]$$

as promised.
In the limit $\Gamma \to 0$, this reduces to

$$\rho_{ee} = \frac{|\Omega|^2}{(\omega_0 - \omega_L)^2}\sin[(\omega_0 - \omega_L)t/2]$$

which is the familiar form of Rabi flopping in a coherent two-level system with $\Omega << (\omega_0 - \omega_L)$. We note that these oscillations were damped due to the presence of spontaneous emission.

b) Show that the solution of these equations to lowest order in $|\Omega|$ in the limit $|\Omega|t \ll 1$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{1}{4}|\Omega|^2 t^2\,, \tag{4}$$

irrespective of the values of $(\omega_0 - \omega_L)$ and Γ. Why?

Note that the limits are identical to those of part a), so we can just Taylor expand our solution for a) to second order in t. All we need are

$$\begin{aligned}\cos(x) &= 1 - \frac{x^2}{2!} + O(x^4)\\ e^x &= 1 + x + \frac{x^2}{2!} + O(x^3)\end{aligned}$$

and the result follows trivially.

A second option is to write out a second order Taylor expansions for ρ_{ee} and ρ_{ge},

$$\rho_{ee} \approx a_0 + a_1 t + \frac{a_2}{2} t^2 \;\Rightarrow\; \dot{\rho_{ee}} \approx a_1 + a_2 t$$
$$\rho_{ge} \approx b_0 + b_1 t + \frac{b_2}{2} t^2 \;\Rightarrow\; \dot{\rho_{ge}} \approx b_1 + b_2 t\,,$$

plug these back into the given differential equations, and match powers of t. Both approaches give the promised result. For short times, one expects the result to be independent of the detuning as there aren't enough cycles of the drive field for the atom to accurately determine the frequency. Likewise, the evolution is completely coherent, so Γ doesn't come into the picture either.

3. **Single photon generation from resonance flourescence of a single atom**.

Note for the curious. This problem is nicely and fully covered in Rep. Prog. Phys., Vol 43, 1980, *Non-classical effects in the statistical properties of light*, by Rodney Loudon.

a) Quantum Regression Theorem

Define $A(t) \equiv \sigma_+(t)\sigma_-(t)$.
Note that $\langle A(t)\rangle = \rho_{ee}(t)$ (the $A(t)$ operator is non-zero only for state $|e\rangle$).
Thus, by eq 11,

$$\langle A(t)\rangle = \alpha_1(t) + \alpha_2(t)\langle \rho_{ge}(0)\rangle + \alpha_3(t)\langle \rho_{eg}(0)\rangle + \alpha_4(t)\langle \rho_{ee}(0)\rangle$$

This is precisely a decomposition of $\langle A(t)\rangle$ of the form in eq 9, where $A_1 = 1$, $A_2 = |g\rangle\langle e|$, $A_3 = |e\rangle\langle g|$, $A_4 = |e\rangle\langle e|$.

Now examine the numerator of our expression for $g^{(2)}(\tau)$.

$$\langle \sigma_+(0)\sigma_+(\tau)\sigma_-(\tau)\sigma_-(0)\rangle = \langle \sigma_+(0)A(t)\sigma_-(0)\rangle$$

By the Onsager-Lax theorem, this is equal to

$$\sum_k \alpha_k(t)\langle \sigma_+(0)A_k(0)\sigma_-(0)\rangle$$

Note that from our earlier definitions of the A_k operators, the expectation value above is non-zero for only the $k = 1$ term. Thus, we have reduced the numerator of eq

12 to

$$\alpha_1(t)\langle\sigma_+(0)\sigma_-(0)\rangle.$$

If we assume that we are operating at steady state, then without any a-priori information about the state of the system, the density matrix at time 0 should be the same as at time ∞. Thus we can assert that

$$\langle\sigma_+(0)\sigma_-(0)\rangle = \langle\sigma_+(\infty)\sigma_-(\infty)\rangle = \alpha_1(\infty)$$

So our expression for $g^{(2)}(\tau)$ is

$$\frac{\alpha_1(\infty)\alpha_1(\tau)}{\alpha_1(\infty)^2} = \frac{\alpha_1(\tau)}{\alpha_1(\infty)}$$

b) Master equation approximation – decay terms only

The given differential equation is fairly simple to solve. Take as an ansatz a solution of the form $x = c_0 + c_1 \exp(c_2 t)$ and plug this into the equation, you can fairly quickly get that the general solution is:

$$\rho_{ee} = \frac{g}{\Gamma} + c_1 \exp(-\Gamma t)$$

Where c_1 is determined by the initial conditions. Solving for $c1$ based on $\rho_{ee}(0)$, and rewriting in the form of eq 11, we have

$$\rho_{ee}(t) = \frac{g}{\Gamma}(1 - \exp(-\Gamma t)) + \rho_{ee}(0)\exp(-\Gamma t)$$

From which we can read out α_1, and get

$$g^{(2)}(\tau) = 1 - \exp(-\Gamma t)$$

c) The expression for $g^{(2)}(\tau)$ obtained from the weak excitation Bloch Equations is simply

$$g^{(2)}(\tau) = 1 + \exp(-\Gamma t) - 2\cos(\delta t)\exp(-\Gamma t/2)$$

This has been plotted for a number of values of δ/Γ in the following figure, taken from the Loudon reference given above.

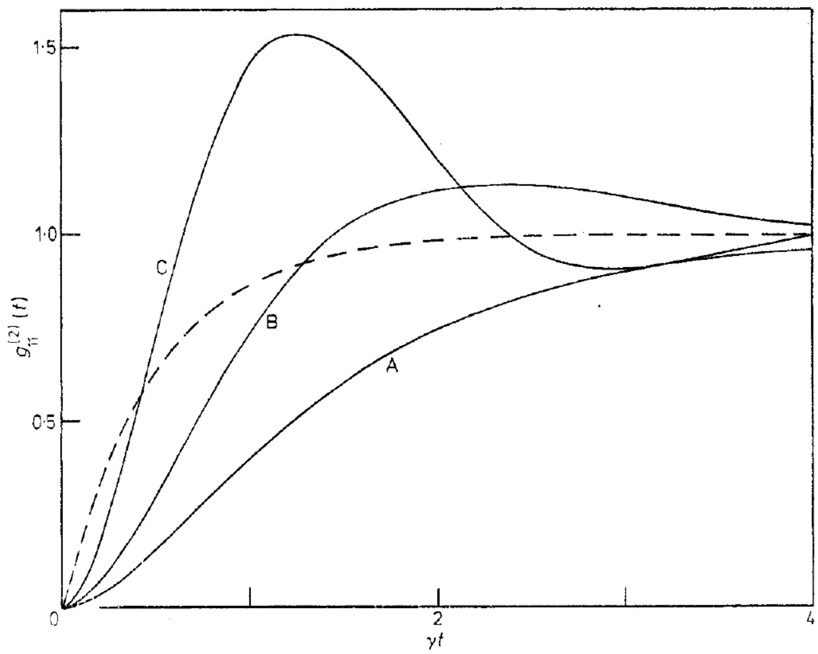

Figure 5. Time dependence of the degree of second-order coherence. Broken curve: the rate-equation result (4.15). Full curves: the weak-beam, zero collision broadening result (4.29) for A, $\Delta=0$; B, $\Delta=\gamma$ and C, $\Delta=2\gamma$.

d) The meaning of $g^{(2)}(\tau)$

$g^{(2)}(\tau)$ compares the probability of detecting two photons, spaced by time τ, with the square of the probability of finding a single photon. For totally independent photons, ie a classical field or coherent state, these two quantities are equal and $g^{(2)}(\tau) = 1$. For short times, in the above computed $g^{(2)}(\tau)$ functions, we find that the probability of two-photon events is supressed, which is "photon anti-bunching".