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Assignment #5: Solutions

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1. **Unravelings of spontaneous emission dynamics**.

(a) For $H = I$ and $L = \sqrt{\Gamma}\sigma_- = \sqrt{\Gamma}(X - iY)/2$.

$$\begin{aligned}\dot{\rho} &= -\frac{i}{\hbar}[H,\rho] + \left[L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right] \\ &= \Gamma\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right]\end{aligned}$$

For $H = -\Gamma\hbar Y/4$ and $L = \sqrt{\Gamma}(I + X - iY)/2$.

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \left[L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right]$$

$$-\frac{i}{\hbar}[H,\rho] = \Gamma\left[\frac{1}{4}\sigma_+\rho - \frac{1}{4}\sigma_-\rho - \frac{1}{4}\rho\sigma_+ + \frac{1}{4}\rho\sigma_-\right]$$

$$\left[L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right] = \Gamma\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_- - \frac{1}{4}\sigma_+\rho + \frac{1}{4}\sigma_-\rho + \frac{1}{4}\rho\sigma_+ - \frac{1}{4}\rho\sigma_-\right]$$

Thus, $\dot{\rho} = \Gamma\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right]$. Dynamics are exactly the same.

(b) Giving $H = \Gamma Y/4$ and $L = \sqrt{\Gamma}(-I + X - iY)/2$, we also get $\dot{\rho} = \Gamma\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right]$. This is equivalent to the optical Bloch equations.

More generally (and less trivially), one can construct an infinite number of equivalent unravelings, following this procedure (among others). Start with the operator sum representation (OSR) for spontaneous emission, which has these two operation elements:

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \quad (1)$$

$$E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}, \quad (2)$$

where γ parameterizes the decay rate. The master equation formed from this OSR is the usual form for spontaneous emission. Specifically, the Hamiltonian from the damping,

$H = (E_0 + E_0^\dagger)/2$ to leading order in γ, is identity, and the jump operator $L_1 = \frac{1}{\sqrt{\gamma}} E_1$ is σ_-, to leading order in $\sqrt{\gamma}$. Thus

$$\dot{\rho} = -i[H, \rho] + \sum_{k \neq 0} L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k \tag{3}$$

is the usual master equation in Lindblad form.

However, as discussed in lecture, there is a unitary degree of freedom in the OSR, such that

$$F_j = \sum_k U_{jk} E_k \tag{4}$$

also give a valid set of operation elements, for any unitary matrix with elements U_{jk}, describing an identical operation. Specifically, if we take U to be an $O(\sqrt{\gamma})$ rotation about axis $\hat{n}$,

$$U = e^{-i \frac{\sqrt{\gamma}}{2} \hat{n} \cdot \vec{\sigma}} = R_{\hat{n}}(\sqrt{\gamma}), \tag{5}$$

where $\vec{\sigma}$ is a vector of Pauli matrices and $\hat{n}$ is an arbitrary unit vector, then the resulting $\{F_j\}$ operators can be directly translated to give master equations in Lindblad form which appear different to the usual Bloch equations, but actually have exactly the same dynamics. Specifically, the result $H = -\Gamma Y/4$ and $L = \sqrt{\Gamma}(I + X - iY)/2$ comes from choosing $\hat{n} = \hat{y}$, and following the same procedure as in the previous paragraph.

What is the physical interpretation? Note that E_0 represents the transformation which occurs when no photon is emitted from an atom in time $\sim \gamma$, while E_1 corresponds to what happens when a single photon is emitted. U represents the basis used by the environment in its "measurement" of the photon emitted from the atom. Thus, when $U = R_y(\sqrt{\gamma})$ the action of the environment may be interpreted as observing emitted photons in a rotated basis, say with through a beamsplitter of partial reflectivity $\sim \gamma$, with detectors on both outputs of the beamsplitter. This is a form of homodyne measurement, so the environment collapses the state of the atom not in the basis of distinct photon numbers, but rather, of distinct quadrature component values.

2. **Two-bit code for spontaneous emission errors**.

(a) For an input state $|\psi_1\rangle = (a|0\rangle + b|1\rangle)$, (We take a and b to be real to simplify the problem)

$$|\psi_1\rangle\langle\psi_1| = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

The fidelity of ρ_1 with respect to $|\psi_1\rangle$ is

$$\begin{aligned} F_1(t) = F(|\psi_1\rangle, \rho_1) &= \sqrt{\langle\psi_1|\rho_1|\psi_1\rangle} = \sqrt{(a^2 + b^2\sqrt{1-\gamma})^2 + (ab\sqrt{\gamma})^2} \\ &= \sqrt{a^4 + a^2 b^2(\gamma + 2\sqrt{1-\gamma}) + b^4(1-\gamma)} \end{aligned}$$

In the case, $|\psi_1\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$,

$$F_1(t) = F(|\psi_1\rangle, \rho_1) = \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1-\gamma}}$$

Plot $F_1(t) = F(|\psi_1\rangle, \rho_1)$ as a function of t for $\gamma = 1 - e^{-t/T_1}$, the fidelity approaches $1/\sqrt{2}$

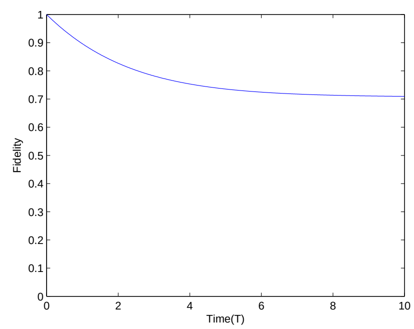

FIG. 1: Fidelity vs. time

(b) Rewrite $F_1(t)$ using $x = a^2 = 1 - b^2$

$$F_1(x) = \sqrt{2x^2(1-\gamma-\sqrt{1-\gamma}) + x(3\gamma - 2 + 2\sqrt{1-\gamma}) + (1-\gamma)}$$

In the range 0≤ γ ≤1 and 0≤ x ≤1, $F_1(x)$ is an increasing function. When $x = 0$, i.e. $|\phi\rangle = |1\rangle$, it has the lowest fidelity at all times.

$$F_{min}(t) = \sqrt{1-\gamma} = e^{\frac{-t}{2T_1}}$$

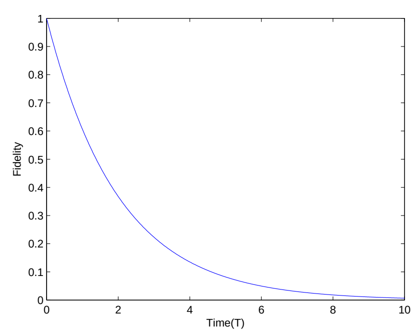

FIG. 2: Fidelity vs. time

(c) Let the input state $|\psi\rangle = a|0_L\rangle + b|1_L\rangle = a|01\rangle + b|10\rangle$

$$\begin{aligned}
\rho' = \mathcal{E}(|\psi\rangle) &= \sum_{j,k=\{0,1\}} (E_j \otimes E_k)\, |\psi\rangle\langle\psi|\, (E_j \otimes E_k)^\dagger \\
&= \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & (1-\gamma)a^2 & (1-\gamma)ab & 0 \\ 0 & (1-\gamma)ab & (1-\gamma)b^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \gamma|00\rangle\langle 00| + (1-\gamma)|\psi\rangle\langle\psi|.
\end{aligned}$$

(d) The fidelity of ρ' with respect to $|\psi\rangle = a|01\rangle + b|10\rangle$ is

$$F(|\psi\rangle, \rho') = \sqrt{1-\gamma} = e^{\frac{-t}{2T_1}}$$

which is the same as (b). Thus the same plot.

(e) When we project the output state into the space orthogonal to $|00\rangle$ and keep only the case when we do not obtain $|00\rangle$, the resulting state is $|\psi\rangle$. Conditioned on not obtaining $|00\rangle$, the fidelity of the post-selected state with respect to $|\psi\rangle$ is 1.

3. **Rayleigh and Thomson scattering using two different interaction Hamiltonians**.

(a) Transition matrix element calculation.

$$\mathbf{A}(\mathbf{0}) = \int d^3k \sum_{\varepsilon} \sqrt{\frac{\hbar}{2\varepsilon_0 L^3 \omega}} \left[a_\varepsilon(\mathbf{k}) + a_\varepsilon^\dagger(\mathbf{k})\right] \boldsymbol{\varepsilon}$$

$$\mathbf{E}_\perp(\mathbf{0}) = \int d^3k \sum_{\varepsilon} \mathrm{i} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 L^3}} \left[a_\varepsilon(\mathbf{k}) - a_\varepsilon^\dagger(\mathbf{k})\right] \boldsymbol{\varepsilon}$$

$$\mathcal{T}_{fi} = \langle\psi_f|H_I|\psi_i\rangle + \lim_{\epsilon\to 0^+} \sum_j \frac{\langle\psi_f|H_I|\psi_j\rangle\langle\psi_j|H_I|\psi_i\rangle}{E_i - E_j + \mathrm{i}\epsilon} = \mathcal{T}_{fi}(1) + \mathcal{T}_{fi}(2)$$

The intermediate state of the scattering process is either $|b;0\rangle$ or $|b;\mathbf{k}\boldsymbol{\varepsilon},\mathbf{k}'\boldsymbol{\varepsilon}'\rangle$.
For electrodipole Hamiltonian $H_I' = -\mathbf{d}\cdot\mathbf{E}_\perp(\mathbf{0})$.

$$\langle b;0|H_I'|a;\mathbf{k}\boldsymbol{\varepsilon}\rangle = -\mathrm{i}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 L^3}}\langle b|\mathbf{d}\cdot\boldsymbol{\varepsilon}|a\rangle$$

$$\langle a;\mathbf{k}'\boldsymbol{\varepsilon}'|H_I'|b;0\rangle = \mathrm{i}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 L^3}}\langle a|\mathbf{d}\cdot\boldsymbol{\varepsilon}'|b\rangle$$

$$\langle b; \mathbf{k}\boldsymbol{\varepsilon}, \mathbf{k}'\boldsymbol{\varepsilon}'|H_I'|a; \mathbf{k}\boldsymbol{\varepsilon}\rangle = \mathrm{i}\sqrt{\frac{\hbar\omega'}{2\varepsilon_0 L^3}}\langle b|\mathbf{d}\cdot\boldsymbol{\varepsilon}'|a\rangle$$

$$\langle a; \mathbf{k}'\boldsymbol{\varepsilon}'|H_I'|b; \mathbf{k}\boldsymbol{\varepsilon}, \mathbf{k}'\boldsymbol{\varepsilon}'\rangle = -\mathrm{i}\sqrt{\frac{\hbar\omega'}{2\varepsilon_0 L^3}}\langle a|\mathbf{d}\cdot\boldsymbol{\varepsilon}|b\rangle$$

$$\mathcal{T}_{fi}'(1) = 0$$

$$\mathcal{T}_{fi}'(2) = \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\sqrt{\omega\omega'}}\sum_b(\omega\omega')\left[\frac{\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}|a\rangle}{\hbar(\omega-\omega_{ba})} - \frac{\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|a\rangle}{\hbar(\omega'+\omega_{ba})}\right]$$

For the Coulomb-gauge Hamiltonian $-\frac{q}{m}\mathbf{p}\cdot\mathbf{A}(\mathbf{0}) + \frac{q^2}{2m} + \mathbf{A}^2(\mathbf{0})$

$$\mathcal{T}_{fi}(1) = \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\sqrt{\omega\omega'}}\frac{1}{m}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'$$

$$\mathcal{T}_{fi}(2) = \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\sqrt{\omega\omega'}}\frac{1}{m^2}\sum_b\left[\frac{\langle a|\mathbf{p}\cdot\boldsymbol{\varepsilon}'|b\rangle\langle b|\mathbf{p}\cdot\boldsymbol{\varepsilon}|a\rangle}{\hbar(\omega-\omega_{ba})} - \frac{\langle a|\mathbf{p}\cdot\boldsymbol{\varepsilon}|b\rangle\langle b|\mathbf{p}\cdot\boldsymbol{\varepsilon}'|a\rangle}{\hbar(\omega'+\omega_{ba})}\right]$$

$$< b|\mathbf{p}\cdot\boldsymbol{\varepsilon}|a >= \mathrm{i}m\omega_{ba} < b|\mathbf{r}\cdot\boldsymbol{\varepsilon}|a >$$

$$\mathcal{T}_{fi}(2) = \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\sqrt{\omega\omega'}}\sum_b(\omega_{ba}^2)\left[\frac{\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}|a\rangle}{\hbar(\omega-\omega_{ba})} - \frac{\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|a\rangle}{\hbar(\omega'+\omega_{ba})}\right]$$

For Rayleigh and Thomson Scattering, $\omega = \omega'$

$$\begin{aligned}\mathcal{T}_{fi}' - \mathcal{T}_{fi}(2) &= \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\omega}\sum_b(\omega^2-\omega_{ba}^2)\left[\frac{\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}|a\rangle}{\hbar(\omega-\omega_{ba})} - \frac{\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|a\rangle}{\hbar(\omega+\omega_{ba})}\right] \\ &= \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\omega}\sum_b\frac{1}{\hbar}\left[(\omega+\omega_{ba})\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}|a\rangle - (\omega-\omega_{ba})\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|a\rangle\right] \\ &= \mathcal{T}_{fi}(1)\end{aligned}$$

For Rayleigh scattering $\omega \ll \omega_{ba}$

$$\mathcal{T}_{fi}' = -\frac{q^2}{2\varepsilon_0 L^3}\omega\sum_b\frac{\langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}|a\rangle + \langle a|\mathbf{r}\cdot\boldsymbol{\varepsilon}|b\rangle\langle b|\mathbf{r}\cdot\boldsymbol{\varepsilon}'|a\rangle}{\omega_{ba}}$$

For Thomson scattering $\omega \gg \omega_{ba}$

$$\mathcal{T}_{fi}(1) = \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\omega}\frac{1}{m}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'$$

$$\mathcal{T}'_{fi} = \mathcal{T}_{fi}(1) + \mathcal{T}_{fi}(2)$$

$\omega \gg \omega_{ba} \Rightarrow \mathcal{T}'_{fi} \gg \mathcal{T}_{fi}(2)$, i.e. $\mathcal{T}_{fi}(1) \gg \mathcal{T}_{fi}(2)$

$$\mathcal{T}_{fi} = \mathcal{T}_{fi}(1) = \frac{q^2\hbar}{2\varepsilon_0 L^3}\frac{1}{\omega}\frac{1}{m}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'$$

(b) Thomson scattering cross section calculations
The transition probability per unit time and per unit solid angle is

$$\begin{aligned}\frac{\delta w_{fi}}{\delta\boldsymbol{\Omega}'} &= \frac{2\pi}{\hbar}\left(\frac{q^2}{2m}\frac{\hbar}{\varepsilon_0 L^3\omega}\right)^2(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')^2\frac{L^3}{8\pi^3}\frac{(\hbar ck)^2}{\hbar^3 c^3}\\ &= \frac{e^4}{m^2c^3L^3}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')^2\end{aligned}$$

where $e^2 = \frac{q^2}{4\pi\varepsilon_0}$. If we divide by the photon flux, which is equal to c/L^3, the differential cross section is

$$\frac{d\sigma}{d\boldsymbol{\Omega}'} = r_0^2(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')^2$$

To find the total cross section, sum $(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}')^2$ over the two polarizations orthogonal to $\mathbf{k}'$ and to make the angular average.
Thus the total cross section for Thomson scattering is $\frac{8\pi}{3}r_0^2$, where r_0 is the classical electron radius.