

Assignment #6

Due: Wednesday, April 1st 2009
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 Office Hours: Monday, March 30th and Tuesday, March 31st 6pm - 8pm

1. Long-range (Van der Waals) interaction between ground-state atoms

- (a) (2 Points) What is the first non-vanishing term in the series for the perturbed ground state energy of the system?

Solution: Using standard perturbation theory, the series for the perturbed ground state energy of the system is

$$\Delta E_{gg} = V_{gg} + \sum_{i \neq g} \frac{|V_{ig}|^2}{E_g^0 - E_i^0} + \text{higher-order terms}$$

Take the z -axis to be along the internuclear separation vector \vec{R} . The perturbation Hamiltonian becomes

$$H_{E1} = e^2 \frac{(x_a x_b + y_a y_b - 2z_a z_b)}{R^3}$$

and

$$\Delta E_g = e^2 \frac{\langle g_a g_b | x_a x_b + y_a y_b - 2z_a z_b | g_a g_b \rangle}{R^3} + e^4 \sum_{i_a i_b} \frac{|\langle i_a i_b | \frac{x_a x_b + y_a y_b - 2z_a z_b}{R^3} | g_a g_b \rangle|^2}{(E_g^a + E_g^b) - (E_i^a + E_i^b)} + \text{higher-order terms}$$

The first-order term in this perturbation series vanishes regardless of the parity of the ground state because H_{E1} is an odd-parity operator [1].

Thus, the first non-vanishing term in the perturbation series is the second-order term which varies like $\frac{1}{R^6}$.

- (b) (5 Points) Express the result from (a) in terms of oscillator strengths.

Solution: We need to evaluate the second-order term:

$$\begin{aligned} \Delta E_{g_a g_b} &= \sum_{n_a, l_a, n_b, l_b} \sum_{m_a, m_b} e^4 \frac{|\langle i_a i_b | \frac{x_a x_b + y_a y_b - 2z_a z_b}{R^3} | g_a g_b \rangle|^2}{(E_g^a + E_g^b) - (E_i^a + E_i^b)} \\ &= e^4 \sum_{n_a, l_a, n_b, l_b} \frac{\frac{1}{R^6} [\sum_{m_a m_b} |\langle i_a i_b | x_a x_b + y_a y_b - 2z_a z_b | g_a g_b \rangle|^2]}{(E_g^a + E_g^b) - (E_i^a + E_i^b)} \end{aligned}$$

Notice that the sum over the upper states has been explicitly separated into sums over quantum numbers n and l and sums over the orientational quantum number m .

Next the sum over the upper state orientational quantum numbers m_a and m_b is calculated

$$\begin{aligned} S &\equiv \sum_{m_a m_b} |\langle i_a i_b | x_a x_b + y_a y_b - 2z_a z_b | g_a g_b \rangle|^2 \\ &= \sum_{m_a m_b} \langle x_a \rangle \langle x_b \rangle \langle x_a \rangle^* \langle x_b \rangle^* + \langle x_a \rangle \langle x_b \rangle \langle y_a \rangle^* \langle y_b \rangle^* - 2 \langle x_a \rangle \langle x_b \rangle \langle z_a \rangle^* \langle z_b \rangle^* \\ &\quad + \langle y_a \rangle \langle y_b \rangle \langle x_a \rangle^* \langle x_b \rangle^* + \langle y_a \rangle \langle y_b \rangle \langle y_a \rangle^* \langle y_b \rangle^* - 2 \langle y_a \rangle \langle y_b \rangle \langle z_a \rangle^* \langle z_b \rangle^* \\ &\quad - 2 \langle z_a \rangle \langle z_b \rangle \langle x_a \rangle^* \langle x_b \rangle^* - 2 \langle z_a \rangle \langle z_b \rangle \langle y_a \rangle^* \langle y_b \rangle^* + 4 \langle z_a \rangle \langle z_b \rangle \langle z_a \rangle^* \langle z_b \rangle^* \end{aligned}$$

To save space above, I have used some shorthand notation:

$$\langle x_a \rangle \equiv \langle n_a, l_a, m_a | x_a | g_a \rangle \text{ and } \langle x_b \rangle \equiv \langle n_b, l_b, m_b | x_b | g_b \rangle$$

In other words, $\langle x_a \rangle$ represents a matrix element not an expectation value.

Rearranging we have

$$\begin{aligned} S &= \left(\sum_{m_a} \langle x_a \rangle \langle x_a \rangle^* \right) \left(\sum_{m_b} \langle x_b \rangle \langle x_b \rangle^* \right) + \left(\sum_{m_a} \langle x_a \rangle \langle y_a \rangle^* \right) \left(\sum_{m_b} \langle x_b \rangle \langle y_b \rangle^* \right) \\ &\quad - 2 \left(\sum_{m_a} \langle x_a \rangle \langle z_a \rangle^* \right) \left(\sum_{m_b} \langle x_b \rangle \langle z_b \rangle^* \right) + \left(\sum_{m_a} \langle y_a \rangle \langle x_a \rangle^* \right) \left(\sum_{m_b} \langle y_b \rangle \langle x_b \rangle^* \right) \\ &\quad + \left(\sum_{m_a} \langle y_a \rangle \langle y_a \rangle^* \right) \left(\sum_{m_b} \langle y_b \rangle \langle y_b \rangle^* \right) - 2 \left(\sum_{m_a} \langle y_a \rangle \langle z_a \rangle^* \right) \left(\sum_{m_b} \langle y_b \rangle \langle z_b \rangle^* \right) \\ &\quad - 2 \left(\sum_{m_a} \langle z_a \rangle \langle x_a \rangle^* \right) \left(\sum_{m_b} \langle z_b \rangle \langle x_b \rangle^* \right) - 2 \left(\sum_{m_a} \langle z_a \rangle \langle y_a \rangle^* \right) \left(\sum_{m_b} \langle z_b \rangle \langle y_b \rangle^* \right) \\ &\quad + 4 \left(\sum_{m_a} \langle z_a \rangle \langle z_a \rangle^* \right) \left(\sum_{m_b} \langle z_b \rangle \langle z_b \rangle^* \right) \end{aligned}$$

We expect that the ‘‘off-diagonal’’ terms in the above sum will ‘‘average’’ to zero. Let us calculate this explicitly for the case of upper states $|i\rangle$ which have $l = 1$ (i.e. $|P\rangle$ states) and lower states $|g\rangle$ which have $l = 0$ (i.e. $|S\rangle$ states). To do so we express the dipole operator and the angular dependences of the $|P\rangle$ and $|S\rangle$ states in terms of spherical harmonics

$$x = r \sqrt{\frac{4\pi}{3}} \frac{1}{\sqrt{2}} (Y_1^{-1} - Y_1^1)$$

$$y = r \sqrt{\frac{4\pi}{3}} \frac{1}{\sqrt{2}} (Y_1^{-1} + Y_1^1)$$

$$z = r \sqrt{\frac{4\pi}{3}} Y_1^0$$

Two facts:

- i. The angular dependence of a state $|l, m\rangle$ is given by Y_l^m .
- ii. The orthogonality condition for spherical harmonics, $\int (Y_{l_1}^{m_1})^* Y_{l_2}^{m_2} d\Omega = \delta_{l_1 l_2} \delta_{m_1 m_2}$.

Using these facts we obtain the following results for our matrix elements (Of course, the calculation is made simpler by the fact that $|S\rangle$ states have no angular dependence because $Y_0^0 = \frac{1}{\sqrt{4\pi}}$. Another way: I chose the $|S\rangle$ states for a reason!)

$$\begin{aligned} \langle l=1, m=+1|x|l=0, m=0\rangle &= \frac{-\langle r\rangle}{\sqrt{6}} & \langle l=1, m=+1|y|l=0, m=0\rangle &= \frac{1\langle r\rangle}{\sqrt{6}} \\ \langle l=1, m=0|x|l=0, m=0\rangle &= 0 & \langle l=1, m=0|y|l=0, m=0\rangle &= 0 \\ \langle l=1, m=-1|x|l=0, m=0\rangle &= \frac{\langle r\rangle}{\sqrt{6}} & \langle l=1, m=-1|y|l=0, m=0\rangle &= \frac{1\langle r\rangle}{\sqrt{6}} \\ \langle l=1, m=+1|z|l=0, m=0\rangle &= 0 \\ \langle l=1, m=0|z|l=0, m=0\rangle &= \frac{\langle r\rangle}{\sqrt{3}} \\ \langle l=1, m=-1|z|l=0, m=0\rangle &= 0 \end{aligned}$$

Our sum over orientations S becomes

$$\begin{aligned} S &= \left(\frac{\langle r_a\rangle^2}{3}\right) \left(\frac{\langle r_b\rangle^2}{3}\right) + 0 + 0 + 0 + \left(\frac{\langle r_a\rangle^2}{3}\right) \left(\frac{\langle r_b\rangle^2}{3}\right) + 0 + 0 + 0 + 4 \left(\frac{\langle r_a\rangle^2}{3}\right) \left(\frac{\langle r_b\rangle^2}{3}\right) \\ &= 6 \left(\frac{\langle r_a\rangle^2}{3}\right) \left(\frac{\langle r_b\rangle^2}{3}\right) \end{aligned}$$

Thus we have the result

$$\Delta E_{g_a g_b} = -\frac{e^4}{R^6} \sum_{n_a, l_a, n_b, l_b} \frac{6 \left(\frac{\langle r_a\rangle^2}{3} \frac{\langle r_b\rangle^2}{3}\right)}{(E_i^a - E_g^a) + (E_i^b - E_g^b)}$$

Now we introduce oscillator strengths: $f_{ig} \equiv \frac{2m_e \omega_{ig}}{\hbar} |\langle m_i|x|m_g\rangle|^2$. Note that the mass, m_e , is the electron mass. Afterall, it is the oscillator.

The choice of x, y , or z in our definition of oscillator strength represents an artificial choice of axis; we should obtain the same f_{ig} if we sum over upper m states and average over lower m states.

Using the dipole operator matrix elements derived above for $|P\rangle \rightarrow |S\rangle$ transitions, we obtain (In this case there is no need to average over lower m states because there is only one.)

$$\sum_{m_i} |\langle m_i|x|m_g\rangle|^2 = \sum_{m_i} |\langle m_i|y|m_g\rangle|^2 = \sum_{m_i} |\langle m_i|z|m_g\rangle|^2 = \frac{\langle r\rangle^2}{3}$$

Therefore,

$$f_{ig} \equiv \frac{2m_e \omega_{ig}}{\hbar} \frac{\langle r\rangle^2}{3} \Rightarrow \frac{\langle r\rangle^2}{3} = \frac{\hbar}{2m_e} \frac{f_{ig}}{\omega_{ig}}$$

Finally,

$$\begin{aligned}\Delta E_{g_a g_b} &= -\frac{3}{2} \frac{e^4 \hbar^2}{R^6 m_e^2} \sum_{n_a, l_a, n_b, l_b} \frac{\frac{f_{ig}^a}{\omega_{ig}^a} \frac{f_{ig}^b}{\omega_{ig}^b}}{(E_i^a - E_g^a) + (E_i^b - E_g^b)} \\ &= -\frac{C_6}{R^6}\end{aligned}$$

where

$$C_6 = \frac{3e^4 \hbar}{2m_e^2} \sum_{n_a, l_a, n_b, l_b} \frac{f_{ig}^a f_{ig}^b}{\omega_{ig}^a \omega_{ig}^b (\omega_{ig}^a + \omega_{ig}^b)}$$

- (c) (3 Points) Express your result for C_6 from (b) in terms of polarizabilities $\alpha_g^{(a)}$ and $\alpha_g^{(b)}$. We will make the dominant-level approximation: assume $f_{ig}^a \approx 1$, $f_{ig}^b \approx 1$ for only one transition and drop the sums over excited states.

$$\Delta E_{g_a g_b} = -\frac{3e^4 \hbar}{2m_e^2} \frac{1}{R^6} \frac{1}{\omega_{ig}^a \omega_{ig}^b (\omega_{ig}^a + \omega_{ig}^b)}$$

Recall that the static polarizability of the ground state is defined by

$$\alpha_g \equiv 2e^2 \sum_i \frac{|\langle i|z|g\rangle|^2}{E_i - E_g} = \frac{e^2}{m_e} \sum_i \frac{f_{ig}}{\omega_{ig}^2}$$

Using the dominant level approximation $\Rightarrow \alpha_g \approx \frac{e^2}{m_e \omega_{ig}^2}$ or $\frac{1}{\omega_{ig}} \approx \frac{m_e \omega_{ig} \alpha_g}{e^2}$. As a result we obtain

$$C_6 = \frac{3}{2} \frac{\alpha_g^a \alpha_g^b E_{ig}^a E_{ig}^b}{E_{ig}^a + E_{ig}^b} = \frac{3}{2} \hbar \frac{\alpha_g^a \alpha_g^b \omega_{ig}^a \omega_{ig}^b}{\omega_{ig}^a + \omega_{ig}^b}$$

Note that this result has the correct units of [Energy] \times [Length]⁶ because polarizability has units of [Volume]= [Length]³.

2. Long-range interaction between an excited atom and a ground-state atom

(a) (4 Points) Distinguishable atoms

- i. How does the energy of the state $|i_a g_b\rangle$ change as a function of the separation R for large distances? What about state $|g_a i_b\rangle$?

Solution: With the restriction that each atom is a two-level system we can solve this problem exactly. In keeping with convention we will denote the excited state of the two-level atom by $|e\rangle$ instead of the more general excited state notation $|i\rangle$ used in problem 1.

Choosing our basis set to be $|e_a g_b\rangle$ and $|g_a e_b\rangle$, the Hamiltonian is

$$H = \begin{bmatrix} E_e^a + E_g^b & V_1 \\ V_1^* & E_g^a + E_e^b \end{bmatrix}$$

where V_1 is the matrix element of H_{E1} .

We can change the zero of the energy scale by subtracting $E_g^a + E_g^b$ from the diagonal elements. We can do this without affecting the off-diagonal elements. The result is

$$H = \begin{bmatrix} \Delta E^a & V_1 \\ V_1^* & \Delta E^b \end{bmatrix}$$

It is now straightforward to solve for the eigenenergies of E in the presence of the perturbation V_1 .

$$\begin{aligned} \det(H - EI) &= \begin{vmatrix} \Delta E^a - E & V_1 \\ V_1^* & \Delta E^b - E \end{vmatrix} \\ &= E^2 - (\Delta E^a + \Delta E^b)E + (\Delta E^a \Delta E^b - V_1 V_1^*) \\ &= 0 \end{aligned}$$

Finding the roots of this equation gives

$$\begin{aligned} E^\pm &= \frac{1}{2}(\Delta E^a + \Delta E^b) \pm \frac{1}{2}\sqrt{(\Delta E^a + \Delta E^b)^2 - 4((\Delta E^a \Delta E^b) - V_1 V_1^*)} \\ &= \frac{1}{2}(\Delta E^a + \Delta E^b) \pm \frac{1}{2}\sqrt{(\Delta E^a - \Delta E^b)^2 + 4V_1 V_1^*} \\ E_\pm &= \overline{\Delta E} \pm \frac{1}{2}\sqrt{\delta E^2 + 4V_1 V_1^*} \end{aligned}$$

where $\overline{\Delta E} \equiv \frac{1}{2}(\Delta E^a + \Delta E^b)$ and $\delta E \equiv \Delta E^a - \Delta E^b$.

At large distances, $\delta E \gg V_1 V_1^*$ so we can approximate the square root.

$$E^\pm = \overline{\Delta E} \pm \delta E \pm (V_1 V_1^*)/\delta E$$

or

$$\boxed{\begin{aligned} E^+ &= \overline{\Delta E} + V_1 V_1^*/\delta E \\ E^- &= \overline{\Delta E} - V_1 V_1^*/\delta E \end{aligned}}$$

Under this approximation, the correction term $V_1 V_1^* / \delta E$ is the same as the 2nd order perturbation term, and will equal C_6 / R^6 as found in the last problem. Due to the sign of δE , the energies of the states will be pushed apart, so the lower energy state loses energy and the higher energy state will gain energy.

ii. For what separation does perturbation theory become invalid?

Solution: Perturbation theory (or the above approximation) will become invalid when the atoms are close enough that the perturbation term H_{E1} is a significant part of the total Hamiltonian. Specifically, perturbation theory will break down when $\frac{C_6}{R^6} \approx \delta E$. From this we can derive an approximate expression for the inter-nuclear separation at which perturbation theory will fail.

Using the results of Problem 1, we have

$$R^6 = \frac{C_6}{R^6} = \frac{\frac{3}{2} \alpha_g^a \alpha_g^b E_{ig}^a E_{ig}^b}{(E_{ig}^a + E_{ig}^b)(\delta E)}$$

If the two atoms are nearly degenerate, then perturbation theory breaks down when

$$R^6 \approx \frac{3}{2} \alpha_g^2 \frac{E}{2\delta E}, \text{ or } R < \left(\frac{3}{2}\right)^{\frac{1}{6}} (\alpha_g)^{\frac{1}{3}} \left(\frac{E}{2\delta E}\right)^{\frac{1}{6}}$$

Since $(\alpha_g)^{\frac{1}{3}}$ has units of length, this result is reasonable. Also we see that when $\delta E = 0$ ordinary non-degenerate perturbation theory is invalid at all distances, and we have to diagonalize the total Hamiltonian exactly.

(b) (4 Points) Indistinguishable atoms – Calculate the long-range interaction potential curves for the case of one excited atom and one ground state atom.

Solution: In this case the atoms are identical, i.e. $E_g^a = E_g^b$, $E_i^a = E_i^b \Rightarrow \Delta E^a = \Delta E^b = \overline{\Delta E} \equiv \Delta E$, and $\delta E = 0$. We must now diagonalize the matrix

$$H = \begin{bmatrix} 0 & V_1 \\ V_1^* & 0 \end{bmatrix}$$

We will get the standard result that the eigenvectors are not the original basis states $|e_a g_b\rangle$ and $|g_a e_b\rangle$ but symmetric and anti-symmetric combinations thereof

$$\begin{aligned} \Psi_S &= \frac{1}{\sqrt{2}} (|e_a g_b\rangle + |g_a e_b\rangle) \\ \Psi_A &= \frac{1}{\sqrt{2}} (|e_a g_b\rangle - |g_a e_b\rangle) \end{aligned}$$

This result is sensible because we cannot distinguish between atoms a and b so labeling them as such is not physically meaningful.

The eigenvalues of our Hamiltonian are simply $E_A^S = \langle \Psi_A^S | H_{E1} | \Psi_A^S \rangle$.

Evaluating $E_A^S = \langle \Psi_S | H_{E1} | \Psi_S \rangle$,

$$\begin{aligned} E_A^S &= \frac{1}{2} (\langle e_a g_b | \pm \langle g_a e_b |) H_{E1} (|e_a g_b\rangle \pm |g_a e_b\rangle) \\ &= \frac{1}{2} (\langle e_a g_b | H_{E1} | e_a g_b \rangle \pm \langle e_a g_b | H_{E1} | g_a e_b \rangle) \pm (\langle g_a e_b | H_{E1} | e_a g_b \rangle) + (\langle g_a e_b | H_{E1} | g_a e_b \rangle) \\ &= \frac{1}{2} (0 \pm \langle e_a g_b | H_{E1} | g_a e_b \rangle \pm \langle g_a e_b | H_{E1} | e_a g_b \rangle + 0) \\ &= \pm \frac{1}{2} (\langle e_a g_b | H_{E1} | g_a e_b \rangle + \langle g_a e_b | H_{E1} | e_a g_b \rangle) \end{aligned}$$

Since atoms a and b are indistinguishable, we get

$$\begin{aligned} E_A^S &= \pm \frac{e^2}{2R^3} (\langle eg | x_a x_b + y_a y_b - 2z_a z_b | ge \rangle + \langle ge | x_a x_b + y_a y_b - 2z_a z_b | eg \rangle) \\ &= \pm \frac{e^2}{2R^3} (\langle e | x | g \rangle \langle g | x | e \rangle + \langle e | y | g \rangle \langle g | y | e \rangle - 2 \langle e | z | g \rangle \langle g | z | e \rangle \\ &\quad + \langle g | x | e \rangle \langle e | x | g \rangle + \langle g | y | e \rangle \langle e | y | g \rangle - 2 \langle g | z | e \rangle \langle e | z | g \rangle) \\ E_A^S &= \pm (|\langle e | x | g \rangle|^2 + |\langle e | y | g \rangle|^2 - 2|\langle e | z | g \rangle|^2) \end{aligned}$$

Next we must consider angular momentum. When two atoms come together to form a Van der Waals (or long-range) “molecule,” we have two possibilities for the total “molecular” angular momentum. That is, we need to consider the projection of the excited level’s angular momentum along the z -axis of the molecule. If this is ± 1 , corresponding to a Π molecular state, then there will be contributions from the x and y terms. On the other hand, a Σ state will contain z terms with zero projection of angular momentum[2].

Σ states ($L = 0$)	Π states ($L = 1$)
$ \langle x \rangle ^2 = 0$	$ \langle x \rangle ^2 = \frac{\langle r \rangle^2}{6}$
$ \langle y \rangle ^2 = 0$	$ \langle y \rangle ^2 = \frac{\langle r \rangle^2}{6}$
$ \langle z \rangle ^2 = \frac{\langle r \rangle^2}{3}$	$ \langle z \rangle ^2 = 0$
$E_A^S = \mp \frac{2e^2 \langle r^2 \rangle}{3R^3}$	$E_A^S = \pm \frac{e^2 \langle r^2 \rangle}{3R^3}$

Hence we have the results:

$$E_A^S = \mp \frac{C_3^\Sigma}{R^3} \text{ for } \Sigma \text{ states}$$

$$E_A^S = \pm \frac{C_3^\Pi}{R^3} \text{ for } \Pi \text{ states}$$

where $C_3^\Sigma = \frac{2e^2 \langle r \rangle^2}{3}$ and $C_3^\Pi = \frac{1}{2} C_3^\Sigma$.

This behavior is shown in the attached figure. Note that only the symmetric states can radiate via dipole processes. This is indicated in the figure by the wavy lines. The atoms in the antisymmetric states must separate in order to radiate.

- (c) (3 Points) For the case of indistinguishable atoms what is the relation between the spontaneous decay rate of the atom and its long-range interaction coefficient?

Solution: Using Fermi's golden rule we can derive the spontaneous decay rate for an atom: $\Gamma = \frac{4e^2\omega^3}{3\hbar c^3} |\langle e|r|g\rangle|^2$.
Hence

$$C_3^\Sigma = \frac{\hbar\Gamma}{2k^3}$$

Again, only the symmetric states (of Σ and Π) can radiate via dipole processes.

3. Casimir model of the electron

The Casimir "vacuum pressure" pulling together plates infinite in extent and separated by a distance a is $\frac{\pi^2\hbar c}{240a^4}$. Therefore square plates of area $a \times a$ separated by a distance a experience an attractive force

$$F_{Cas} = \frac{\pi^2\hbar c}{240a^2} \quad (1)$$

We know that this is a crude estimate because it is correct only for plates of infinite size.

We can estimate the repulsive force between the plates using Gauss' law. The charge density on each plate is $\sigma = \frac{e}{2a^2}$. Applying Gauss' law and ignoring fringing fields[3]

$$\int E \cdot dS = E(2a^2) = 4\pi q_{in} = 4\pi \frac{e}{2} \quad (2)$$

Thus $E = \frac{\pi e}{a^2}$, and the electrostatic repulsion force is

$$F_{el} = \frac{e}{2} E = \frac{\pi e^2}{2a^2} \quad (3)$$

Equating forces

$$F_{Cas} = F_{el} \Rightarrow \frac{\pi^2\hbar c}{240a^2} = \frac{\pi e^2}{2a^2} \quad (4)$$

yields a value for the fine-structure constant

$$\alpha \equiv \frac{e^2}{\hbar c} = \frac{\pi}{120} \approx \frac{1}{38} \quad (5)$$

The actual value of α is approximately $\frac{1}{137}$ so the Casimir model is not very accurate. Nevertheless, it is surprising that such a crude model comes as close as it does.

[1] There is an exception: if both "atoms" possess space-fixed electric dipole or permanent electric quadrupole moments. This can occur if the atoms are in a strong external field, if the "atoms" are a special type of polyatomic molecule, or if the atoms are in $l \geq 1$ ground states.

- [2] Note that this treatment assumes that L and S are good quantum numbers individually, i.e. when spin-orbit coupling effects are small ($H_{E1} \gg H_{LS}$). If $H_{LS} > H_{E1}$, L and S are no longer good quantum numbers, and one must designate states by the quantum numbers J and M_J ($\mathbf{J} = \mathbf{L} + \mathbf{S}$, $M_J = J_z$).
- [3] One can also argue that since the plates carry the same charge there is no electric field between the two plates. This introduces a factor of 2 which shows up in the end as $\alpha = \frac{\pi}{240}$. Either is acceptable.