Assignment #7

Due: Wednesday, April 8.
TA: Caleb Christensen
Office: 26-261
Email: calchris@mit.edu
Phone: x3-4178 Office Hours: Monday, April 6th and Tuesday, April 7th 6pm - 8pm

1. **Classical Model of the Light Force**
Assume that a hydrogenic atom can be modeled classically as an electron harmonically bound to a nucleus, with a resonant frequency ω_0 and damping coefficient γ. The nucleus is fixed at position $\mathbf{x}_0$ while the electron's position is denoted by $\mathbf{x}$. Now suppose the atom is illuminated with an electromagnetic wave of the form

$$\mathbf{E}(\mathbf{x}, t) = \hat{\epsilon} E_0(\mathbf{x}) \cos(\theta(\mathbf{x}) + \omega t) \tag{1}$$

where $\theta(\mathbf{x})$ is the phase of the wave as a function of position $\mathbf{x}$ at time $t = 0$. The dipole moment of the electron may be written as

$$\mathbf{p}(\mathbf{x}, t) = \hat{p}(u \cos(\theta(\mathbf{x}) + \omega t) - v \sin(\theta(\mathbf{x}) + \omega t) \tag{2}$$

Then the force of the light on the atom is

$$\mathbf{F} = (\mathbf{p} \cdot \hat{\epsilon}) \nabla E(\mathbf{x}, t) \tag{3}$$

(a) **Time averaged force**
Make the dipole approximation that $\mathbf{E}(\mathbf{x}) \approx \mathbf{E}(\mathbf{x}_0)$. Show that the time averaged force is

$$\langle \mathbf{F} \rangle = \frac{1}{2} (\hat{p} \cdot \hat{\epsilon})(u \nabla E_0(\mathbf{x}_0) + v E_0(\mathbf{x}_0) \nabla \theta(\mathbf{x}_0)) \tag{4}$$

This expression is exactly analogous to the quantum-mechanically derived force. The first term is the dipole (stimulated) force, and the second term is the scattering (spontaneous) force.

(b) **The potential picture**
Recalculate the time averaged force on the atom from the instantaneous potential energy of a dipole in an electric field. How does this answer differ from that of 1a? Speculate as to why.

(c) **Dipole moment of electron**
Now we will solve explicitly for the dipole moment of the electron. In complex notation, the equation of motion is

$$m \frac{\partial^2 \mathbf{r}}{\partial t^2} + \gamma \frac{\partial \mathbf{r}}{\partial t} + m \omega_0^2 \mathbf{r} = -e \hat{\epsilon} E_0(\mathbf{x}_0) e^{i(\theta(\mathbf{x}_0) + \omega t)} \tag{5}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$. Solve this equation to find $\mathbf{p} = -e\mathbf{r}$. Substitute the quadrature components of $\mathbf{p}$ into the force equation from part (a) to find that

$$\mathbf{F} = -\frac{e^2}{2m\omega_0} \frac{\delta \nabla E_0^2 + \Gamma E_0^2 \nabla \theta}{4\delta^2 + \Gamma^2} \tag{6}$$

where $\delta = \omega - \omega_0$ and $\Gamma = \gamma / m$. Make the approximation that $\omega \approx \omega_0$.

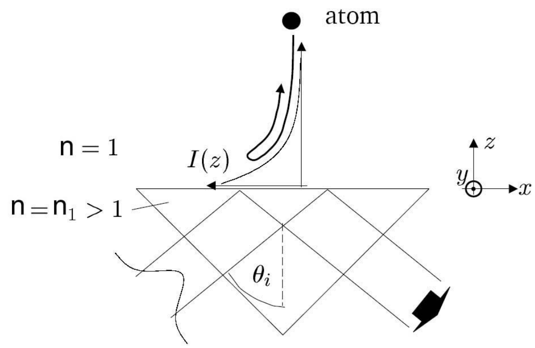
(d) **Force on a two-level atom**
The quantum mechanical expression for the light force on a two-level atom in the low intensity limit is

$$\mathbf{F} \approx -\frac{\hbar\delta\nabla\omega_R^2 + \hbar\Gamma\omega_R^2\nabla\theta}{4\delta^2 + \Gamma^2} \tag{7}$$

where ω_R is the Rabi frequency. Show that if we introduce the oscillator strength f_{fi} between the two levels, we may write

$$\mathbf{F}_{\text{quantum}} = f_{fi}\mathbf{F}_{\text{classical}} \tag{8}$$

2. **An Atomic Trampoline**



It is possible to construct an effective wall for atoms using dipole forces induced by light. A realization of such a wall is the short range repulsive potential formed by an evanescent wave. This wave is produced by total internal reflection of light at a dielectric interface. The interface is mounted horizontally and the atoms are dropped onto it. This problem makes you familiar with the stimulated light force. We have chosen a situation of experimental relevance, where spontaneous emission cannot be fully neglected.
Assume the atoms you are dropping are ideal 2-level systems with mass m. The evanescent wave can be described in units of the Rabi frequency as:

$$\Omega(x) = \Omega_0 e^{-z/l},$$

where Ω_0 is the maximum Rabi frequency at the interface, z is the distance from the interface, and l is the characteristic extension of the evanescent wave (typically $\approx \lambda/2\pi$).
NOTE: Do not consider gravity in 2a through 2e.

(a) For a repulsive interaction, what is the sign of the detuning δ?

(b) Calculate the classical turning point near the interface, z_{tp}, for a bouncing atom in terms of the detuning δ, the line width Γ, the peak Rabi frequency Ω_0, m, and l, if its incident velocity is v. (i.e. v is the velocity with which the atom would hit the interface without the evanescent wave.)

(c) What is the minimum peak Rabi frequency Ω_0 for which the atom would bounce? Express the result in terms of δ, m, l, and Γ as a function of incident velocity v.

(d) Assume that the atom moves perpendicularly to the mirror on a classical trajectory. Find an exact expression for the number N of spontaneously scattered photons per bounce of the form:

$$N = \int g(z)dz,$$

where g is a function of the distance z. (You can assume that $N \ll 1$, so that the spontaneous scattering of photons does not perturb the trajectory.)

(e) Let's assume that $\delta \gg \Gamma$ and $\delta \gg \Omega_0$. In this case, the result of 2d can be simplified, yielding an integral that is exactly solvable. Obtain an expression for N in terms of δ, Γ, v, m, and l.

Useful relations:

$$ln(1+x) \approx x \text{ for } |x| \ll 1$$

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = 2$$

(f) Assume that confinement in the plane parallel to the mirror is somehow provided. Estimate the vertical extension of the ground "bounce" state by using the position-momentum uncertainty relation. Calculate this length for sodium and compare it to $\lambda_{Na}/2\pi$=94 nm. **Hint:** You can treat the problem as an atom moving in a 1-dimensional potential which is linear on one side (the gravitational potential) and has an infinitely steep wall on the other side (the atom-wall potential).

(g) At what 1-dimensional temperature would you expect a significant population in this ground state in the case of "bouncing" sodium atoms? Use this temperature to quickly estimate a vertical trap frequency.
Figure courtesy of D. Schneble, SUNY Stony Brook.
For a relevant experiment refer to PRL **71** p. 3083 (1993).