Solutions to Homework Assignment #7

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1. **Classical Model of the Light Force**
Assume that a hydrogenic atom can be modeled classically as an electron harmonically bound to a nucleus, with a resonant frequency ω_0 and damping coefficient γ. The nucleus is fixed at position $\mathbf{x}_0$ while the electron's position is denoted by $\mathbf{x}$. Now suppose the atom is illuminated with an electromagnetic wave of the form

$$\mathbf{E}(\mathbf{x}, t) = \hat{\epsilon} E_0(\mathbf{x}) \cos(\theta(\mathbf{x}) + \omega t) \tag{1}$$

where $\theta(\mathbf{x})$ is the phase of the wave as a function of position $\mathbf{x}$ at time $t = 0$. The dipole moment of the electron may be written as

$$\mathbf{p}(\mathbf{x}, t) = \hat{p}(u \cos(\theta(\mathbf{x}) + \omega t) - v \sin(\theta(\mathbf{x}) + \omega t) \tag{2}$$

Then the force of the light on the atom is

$$\mathbf{F} = (\mathbf{p} \cdot \hat{\epsilon}) \nabla E(\mathbf{x}, t) \tag{3}$$

(a) **Time averaged force**
Make the dipole approximation that $\mathbf{E}(\mathbf{x}) \approx \mathbf{E}(\mathbf{x}_0)$. Show that the time averaged force is

$$\langle \mathbf{F} \rangle = \frac{1}{2} (\hat{p} \cdot \hat{\epsilon})(u \nabla E_0(\mathbf{x}_0) + v E_0(\mathbf{x}_0) \nabla \theta(\mathbf{x}_0)) \tag{4}$$

This expression is exactly analogous to the quantum-mechanically derived force. The first term is the dipole (stimulated) force, and the second term is the scattering (spontaneous) force.
Answer:
Plug equation 1 and 2 into equation 3, replacing $\mathbf{x}$ with $\mathbf{x}_0$ (since the atom is much smaller than the length scale over which the electric field varies in space):

$$\begin{aligned} \langle \mathbf{F} \rangle \quad = \quad & (\hat{p} \cdot \hat{\epsilon}) \left(u \cos\left(\theta\left(\mathbf{x}_0\right) + \omega t\right) - v \sin\left(\theta\left(\mathbf{x}_0\right) + \omega t\right)\right) \times \\ & \left(\nabla E_0(\mathbf{x}_0) \cos\left(\theta\left(\mathbf{x}_0\right) + \omega t\right) - E_0(\mathbf{x}_0) \nabla \theta\left(\mathbf{x}_0\right) \sin\left(\theta\left(\mathbf{x}_0\right) + \omega t\right)\right) \end{aligned} \tag{5}$$

Cross term that are $\cos(\phi)\sin(\phi)$ will average to zero over a cycle of the laser frequency. In the other terms, the trig functions squared average to $\frac{1}{2}$ and produce the desired equation 4.

(b) **The potential picture**
Recalculate the time averaged force on the atom by first calculating the instantaneous force from the potential. How does this answer differ from that of 1a? Speculate as to why.
Answer:
Using the dipole approximation, the instantanous electrostatic potential energy from an induced electric dipole in the electric field is

$$U = -\frac{1}{2} \mathbf{p} \cdot \mathbf{E}(\mathbf{x}_0, t) \tag{6}$$

substituting in equation 1 and 2 gives

$$U = -\frac{1}{2}\hat{p}\cdot\hat{\epsilon}\left((u\cos(\theta(\mathbf{x}_0)+\omega t) - v\sin(\theta(\mathbf{x}_0)+\omega t)\right)E_0(\mathbf{x}_0)\cos(\theta(\mathbf{x}_0)+\omega t) \quad (7)$$

$$U = -\frac{1}{2}\hat{p}\cdot\hat{\epsilon}E_0(\mathbf{x}_0)\left(u\cos^2(\theta(\mathbf{x}_0)+\omega t) - v\sin(\mathbf{x}_0+\omega t)\cos(\mathbf{x}_0+\omega t)\right) \quad (8)$$

$$U = -\frac{1}{2}\hat{p}\cdot\hat{\epsilon}E_0(\mathbf{x}_0)\left(u\cos^2(\theta(\mathbf{x}_0)+\omega t) - v\frac{1}{2}\sin(2\theta(\mathbf{x}_0)+2\omega t)\right) \quad (9)$$

$$(10)$$

Taking the gradient of the potential gives us a force.

$$\mathbf{F} = -\nabla U \quad (11)$$

$$\mathbf{F} = \frac{1}{2}\hat{p}\cdot\hat{\epsilon}\left(u\cos^2(\theta(\mathbf{x}_0)+\omega t) - v\frac{1}{2}\sin(2\theta(\mathbf{x}_0)+2\omega t)\right)\nabla E_0(\mathbf{x}_0) + \quad (12)$$

$$\frac{1}{2}\hat{p}\cdot\hat{\epsilon}E_0(\mathbf{x})\left(u\sin(2(\theta(\mathbf{x}_0)+2\omega t)) + v\cos(2(\theta(\mathbf{x}_0)+2\omega t))\right)\nabla\theta(\mathbf{x}_0) \quad (13)$$

Averaging over an optical cycle removes all terms except $\cos^2$, which goes to $\frac{1}{2}$.

$$\langle\mathbf{F}\rangle = \frac{1}{4}(\hat{p}\cdot\hat{\epsilon})u\nabla E_0(\mathbf{x}_0) \quad (14)$$

This approach fails to capture the physics of the spontanous force, which acts out of phase with the electric field and cannot be expressed in terms of a conservative potential. The factor of two difference in the stimulated force comes from the fact that u is proportional to $E_0(\mathbf{x}_0)$ (which will be shown later) but is treated as a constant without spatial dependence. This dependence is not consequential in 1a.

(c) **Dipole moment of electron**

Now we will solve explicitly for the dipole moment of the electron. In complex notation, the equation of motion is

$$m\frac{\partial^2\mathbf{r}}{\partial t^2} + \gamma\frac{\partial\mathbf{r}}{\partial t} + m\omega_0^2\mathbf{r} = -e\hat{\epsilon}E_0(\mathbf{x}_0)e^{i(\theta(\mathbf{x}_0)+\omega t)} \quad (15)$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$. Solve this equation to find $\mathbf{p} = -e\mathbf{r}$. Substitute the quadrature components of $\mathbf{p}$ into the force equation from 1a to find that

$$\mathbf{F} = -\frac{e^2}{2m\omega_0}\frac{\delta\nabla E_0^2 + \Gamma E_0^2\nabla\theta}{4\delta^2+\Gamma^2} \quad (16)$$

where $\delta = \omega - \omega_0$ and $\Gamma = \gamma/m$. Make the approximation that $\omega \approx \omega_0$.

Answer:

Since the only vector input to equation 15 is the polarization direction $\hat{\epsilon}$ of the electric field, this will be the unit vector for $\mathbf{r}$ as well. The homogeneous solution decays away and is irrelevant here. The particular solution can be found by using the ansatz:

$$\mathbf{r} = r_0\hat{\epsilon}E_0(\mathbf{x}_0)e^{i(\theta(\mathbf{x}_0)+\omega t)} \quad (17)$$

$$r_0 = \frac{-\frac{e}{m}}{\omega_0^2 - \omega^2 + i\Gamma\omega} \quad (18)$$

Now use $\omega = \delta + \omega_0$ and $\delta \ll \omega_0$ to say $\omega_0^2 - \omega^2 + i\Gamma\omega \approx (-2\delta + i\Gamma)\omega_0$. We will need quadrature components so move the imaginary value to the top of the fraction:

$$\mathbf{p} = -e\mathbf{r} = -\frac{e^2}{m\omega_0}\frac{2\delta+i\Gamma}{4\delta^2+\Gamma^2}\hat{\epsilon}E_0(\mathbf{x}_0)e^{i(\theta(\mathbf{x}_0)+\omega t)} \quad (19)$$

Comparing this to equation 2, we use $e^{i\phi} = \cos(\phi) + i\sin(\phi)$ to identify the real quadrature components u and v (set $\hat{p} = \hat{\epsilon}$):

$$u = -\frac{e^2}{m\omega_0}\frac{2\delta}{4\delta^2+\Gamma^2}E_0(\mathbf{x}_0) \tag{20}$$

$$v = -\frac{e^2}{m\omega_0}\frac{\Gamma}{4\delta^2+\Gamma^2}E_0(\mathbf{x}_0) \tag{21}$$

We can use these in equation 4 and $E\nabla E = \frac{1}{2}\nabla(E^2)$ to get the desired equation 16.

(d) **Force on a two-level atom**

The quantum mechanical expression for the light force on a two-level atom in the low intensity limit is

$$\mathbf{F} \approx -\frac{\hbar\delta\nabla\omega_R^2 + \hbar\Gamma\omega_R^2\nabla\theta}{4\delta^2+\Gamma^2} \tag{22}$$

where ω_R is the Rabi frequency. Show that if we introduce the oscillator strength f_{fi} between the two levels, we may write

$$\mathbf{F}_{\text{quantum}} = f_{fi}\mathbf{F}_{\text{classical}} \tag{23}$$

Answer:

The quantum Rabi frequency is calculated from the matrix elements of a two level atom $|i\rangle$, $|f\rangle$ as

$$\omega_R = \frac{eE_0\langle f|\,\mathbf{r}\cdot\hat{\epsilon}\,|i\rangle}{\hbar} \tag{24}$$

Now E_0 depends on $\mathbf{x}$ so we get$-\frac{e^2|\langle f|\mathbf{r}\cdot\hat{\epsilon}|i\rangle|^2}{\hbar}\frac{\delta\nabla E_0^2+\Gamma E_0^2\nabla\theta}{4\delta^2+\Gamma^2}$

$$\mathbf{F}_{\text{quantum}} = -\frac{e^2\left|\langle f|\,\mathbf{r}\cdot\hat{\epsilon}\,|i\rangle\right|^2}{\hbar}\frac{\delta\nabla E_0^2+\Gamma E_0^2\nabla\theta}{4\delta^2+\Gamma^2} \tag{25}$$

$$= \frac{2m\omega_0\left|\langle f|\,\mathbf{r}\cdot\hat{\epsilon}\,|i\rangle\right|^2}{\hbar}\mathbf{F}_{\text{classical}} \tag{26}$$

$$= f_{fi}\mathbf{F}_{\text{classical}} \tag{27}$$

2. **An Atomic Trampoline**

From Atom-Photon Interactions p.378, V.C.2. Eqn C.40.

$$U_{dipole} = \frac{\hbar\delta}{2}\ln\left[1+\frac{\Omega^2/2}{\delta^2+\gamma^2/4}\right] \tag{28}$$

(a) A positive detuning $\delta > 0$ will result in a repulsive optical dipole potential near the surface.

(b) We can find the classical turning point z_p by applying energy conservation. The incident kinetic energy, far from the surface, $\frac{1}{2}mv_i^2$ must be equal to the potential energy $U(z_p)$ at the turning point. Using Eqn. 28 and substituting $\Omega = \Omega(z) = \Omega_0 e^{-z/l}$ gives

$$\frac{1}{2}mv_i^2 = \frac{1}{2}\hbar\delta\ln\left[1+\frac{\Omega_0^2e^{-2z_p/l}/2}{\delta^2+\gamma^2/4}\right] \tag{29}$$

$$\exp\left(\frac{mv_i^2}{\hbar\delta}\right) = 1+\frac{1}{2}\frac{\Omega_0^2e^{-2z_p/l}}{\delta^2+\gamma^2/4} \tag{30}$$

$$z_p = \frac{l}{2}\ln\left[\frac{\Omega_0^2}{2\delta^2+\gamma^2/2}\frac{1}{\exp\left(mv_i^2/\hbar\delta\right)-1}\right] \tag{31}$$

(c) For an atom to bounce, the peak Rabi frequency Ω_0 must be large enough that $z_p > 0$. Set $z_p = 0$ and solve for Ω_0.

$$
\begin{aligned}
0 &= \frac{l}{2} \ln \frac{\Omega_0^2}{2\delta^2 + \gamma^2/2} \frac{1}{\exp\left(mv_i^2/\hbar\delta\right) - 1} && (32) \\
1 &= \frac{\Omega_0^2}{2\delta^2 + \gamma^2/2} \frac{1}{\exp\left(mv_i^2/\hbar\delta\right) - 1} && (33) \\
\Omega_0^2 &= \left(2\delta^2 + \gamma^2/2\right)\left(\exp\left(mv_i^2/\hbar\delta\right) - 1\right) && (34)
\end{aligned}
$$

(d) To find the an exact expression for the number $N/2$ of spontanously scattered photons per half bounce we integrate the posistion dependent scattering rate $g(z)$ from ∞ to z_p.

$$
\begin{aligned}
N/2 &= \int_{z_p}^{\infty} g(z) dz && (35) \\
N &= 2 \int_{z_p}^{\infty} \frac{\gamma_p(z)}{v(z)} dz && (36)
\end{aligned}
$$

where $\gamma_p(z)$ is the scattering rate and $v(z)$ is the velocity at position z. Since we assume N $\ll 1$, the scattered photons don't affect the trajectory we can calculate $v(z)$ from energy conservation.

$$
\begin{aligned}
U(z) + \frac{1}{2} m v^2(z) &= \frac{1}{2} m v_i^2 && (37) \\
v^2(z) &= v_i^2 - 2U(z)/m && (38)
\end{aligned}
$$

using U(z) from 2b and defining an effective detuning $\alpha^2 = \delta^2 + \gamma^2/4$

$$
v(z) = \sqrt{v_i^2 - (\hbar\delta/m) \ln\left[1 + \frac{\Omega_0 \exp^{-2z/l}}{2\alpha^2}\right]} \qquad (39)
$$

The scattering rate

$$
\begin{aligned}
\gamma_p &= \frac{\gamma\Omega^2}{\gamma^2 + 4\delta^2 + 2\Omega^2} && (40) \\
\gamma_p &= \frac{\gamma}{2} \left(2\frac{\alpha^2}{\Omega^2} + 1\right)^{-1} && (41) \\
\gamma_p(z) &= \frac{\gamma}{2} \left(2\frac{\alpha^2}{\Omega_0^2} e^{2z/l} + 1\right)^{-1} && (42)
\end{aligned}
$$

plugging both of these into the integral gives

$$
N = 2 \int_{z_p}^{\infty} \frac{\gamma}{2} \left(2\frac{\alpha^2}{\Omega_0^2} e^{2z/l} + 1\right)^{-1} \left(v_i^2 - (\hbar\delta/m) \ln\left[1 + \frac{\Omega_0^2 e^{-2z/l}}{2\alpha^2}\right]\right)^{-1/2} dz \qquad (43)
$$

This integral can be simplified without mathematical approximation to the answer in 2e.

(e) Let's assume that $\delta \gg \Gamma$ and $\delta \gg \Omega_0$. In this case, the result of 2d can be simplified, yielding an integral that is easily solvable.
We can approximate the dipole force in this regime as

$$
U \approx \frac{\hbar\Omega^2}{4\delta} \qquad (44)
$$

and the scattering rate as

$$\gamma_p \approx \frac{\gamma\Omega^2}{4\delta^2} \tag{45}$$

which gives a trajectory with velocities

$$v(x) = \sqrt{v_i^2 - \frac{\hbar\Omega_0^2}{2m\delta} e^{-2z/l}} \tag{46}$$

and a total number of scattered photons

$$N \approx 2\int_{z_p}^{\infty} \frac{\gamma\Omega_0^2 e^{-2z/l}}{4\delta^2} \frac{1}{\sqrt{v_i^2 - \frac{\hbar\Omega_0^2}{2m\delta} e^{-2z/l}}} dz. \tag{47}$$

Substituting $y = \frac{\hbar\Omega_0^2}{2mv_i^2\delta} e^{-2z/l}$

$$N \approx \int_1^{\frac{\hbar\Omega_0^2}{2mv_i^2\delta}} \frac{2\gamma\Omega_0^2}{4\delta^2} \frac{1}{\sqrt{v_i^2 - v_i^2 y}} \frac{-mv_i^2\delta l dy}{\hbar\Omega_0^2} dz \tag{48}$$

$$N \approx \frac{\gamma m v_i^2 l}{2\hbar\delta v_i} \int_{\frac{\hbar\Omega_0^2}{2mv_i^2\delta}}^{1} \frac{dy}{\sqrt{1-y}} \tag{49}$$

$$N \approx \frac{\gamma m v_i l}{\hbar\delta} \tag{50}$$

The lower boundry condition on the integral is related to the point of closest approach. To be in the low scattering limit we need to be far (several l) from the surface, so this value approaches zero.

(f) Vertical ground state confinement.

$$U = mgz \quad z > 0 \tag{51}$$

$$U = \infty \quad z < 0 \tag{52}$$

The uncertainty relationship gives us a relationship between Δp and Δz.

$$\Delta p = \hbar/2 \tag{53}$$

$$\Delta p = \frac{\hbar}{2\Delta z} \tag{54}$$

Lets assume that $\bar{p} \sim \Delta p$ and $\bar{z} \sim \Delta z$. Given equipartition of energy

$$mg\Delta z = \frac{\Delta p^2}{2m} \tag{55}$$

$$mg\Delta z = \frac{\hbar^2}{8m\Delta z^2} \tag{56}$$

$$\Delta z^3 = \frac{\hbar^2}{8m^2 g} \tag{57}$$

This gives a vertical range off the mirror of

$$\Delta z = \frac{1}{2}\sqrt[3]{\frac{\hbar^2}{m^2 g}} \tag{58}$$

Plugging in the numbers for ^{23}Na gives a vertical extent of 444 nm (give or take factors of 2 depending on technique). This is much greater then the length scale of the evanscent wave, so the hard wall approximation is justified.

For those who are interested, this problem can actually be solved exactly. The ground state wave function is an Airy function for $z > 0$:

$$\Psi_g(z) = 1.601(\frac{m^2 g}{\hbar^2})^{1/6} Ai(\sqrt[3]{\frac{m^2 g}{\hbar^2}} z - 2.338) \tag{59}$$

From this we can evaluate the expetation value of z numerically to obtain

$$\langle z \rangle = 1.24 \sqrt[3]{\frac{\hbar^2}{m^2 g}} \tag{60}$$

This agrees with our above estimate to within a factor of 2.5.

(g) Temperature of ground state
From equipartition theorem

$$\frac{1}{2} k_B T = \frac{1}{2} m g \Delta x \tag{61}$$

mg for sodium is 27 pK/nm, which gives a temperature of 12 nK. This is pretty cold.
Matching the ground state energy of a harmonic oscillator to the temperature

$$\frac{1}{2} k_B T = \frac{1}{2} \hbar \omega \tag{62}$$

and noting $h/k_B = 4800 Hz/nK$ gives a vertical trap frequency of ∼400 Hz.