Atomic Physics II (8.422) Spring 2009
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Solutions to Assignment 9

1. 1-D Mot

   (a) The atoms in the trap obey a damped harmonic oscillator equation:

   $$\ddot{z} + \gamma\dot{z} + \omega^2 z = 0$$

   The amplitude of oscillation is $z_0 e^{-\gamma t/2}$ and the energy per atom is $E = 1/2\kappa z_0^2 e^{-\gamma t}$, where $\kappa = \omega^2/m$. The cooling rate due to the damping is

   $$\frac{dE}{dt}\mid_{cool} = -\gamma E$$

   There are two heating processes: the random direction of the spontaneous emissions and the temporal randomness of the absorptions. Each event contributes a unit of recoil energy to the heating so

   $$\frac{dE}{dt}\mid_{heat} = 2\Gamma E_{recoil}$$
   $$= 2\Gamma\frac{\hbar^2 k^2}{2M}$$

   Γ is the scattering rate.Equating the two rates, we find an equilibrium energy:

   $$E = \frac{(\hbar k)^2}{2M}\frac{2\Gamma}{\gamma}.$$

   Now we have the equilibrium temperature from the equipartition theorem:

   $$\frac{1}{2}k_B T = \frac{1}{2}M < v^2 > = \frac{1}{2}E \Longrightarrow T = \frac{E}{k_B} = \frac{(\hbar k)^2}{2Mk_B}\frac{2\Gamma}{\gamma}$$
   $$T = \frac{(1.06\times10^{-34}\times\frac{2\pi}{589\times10^{-9}})^2}{2\times23\times1.66\times10^{-27}\times1.38\times10^{-23}}\frac{2\cdot10^7}{10^4}$$
   $$T = 2.4mK$$

   (b) Size of the cloud:

   $$\frac{1}{2}\kappa < z^2 > = \frac{1}{2}M < v^2 > = \frac{1}{2}k_B T$$
   $$< z^2 > = \frac{k_B T}{\kappa} = \frac{k_B T}{M\omega^2}$$
   $$= \frac{(1.38\times10^{-23})(2.4\times10^{-3})}{23\times1.66\times10^{-27}(2\pi)^2(10^3)} =$$
   $$\sqrt{< z^2 >} = 2.0\times10^{-4}m = 200\mu m$$

2. Density Limit in a MOT

   (a) Radiation trapping force:
   Consider two atoms in the trapped cloud, separated by a distance d. There are 6 trapping beams, so the first atom absorbs laser power equal to

   $$P = 6\sigma_L I$$

where $\sigma_L = 6\pi\lambda\!\!\!^{-2}$ is the average absorption cross-section for the laser beams. The first atom then re-radiates this power, and the second atom sees an intensity

$$I_{rad} = \frac{6\sigma_L I}{4\pi d^2}$$

If σ_R is the cross-section for absorption of this light, then the force between the two atoms is

$$F_{rad} = \frac{\sigma_R I_{rad}}{c} = \frac{6\sigma_L\sigma_R I}{4\pi c d^2}$$

Compare Coulomb's Law and Gauss's Law for electrostatics:

$$E = \frac{q}{d^2}; \vec{\nabla}\cdot\vec{E} = 4\pi\rho.$$

Here Gauss' Law gives:

$$\vec{\nabla}\cdot\vec{F}_R = \frac{6\sigma_L\sigma_R I n}{c}$$

(n = atom number density).

(b) Attenuation Force
We derived the repulsive radiation trapping force by considering an atom which absorbs a laser photon and then re-emits it onto another atom with area σ_R. We can also imagine that the first atom, in absorbing the photon, casts a "shadow" of area σ_L in the laser field. The intensity imbalance drives the atom inward, so if we replace σ_R with $-\sigma_L$, we get a divergence equation for the attenuation force:

$$\vec{\nabla}\cdot\vec{F}_A = \frac{-6\sigma_L^2 I n}{c}.$$

(c) At maximum density,

$$\begin{gathered}
\vec{\nabla}\cdot\vec{F}_{total} = 0\\
\vec{\nabla}\cdot\vec{F}_R + \vec{\nabla}\cdot\vec{F}_A + \vec{\nabla}\cdot\vec{F}_T = 0\\
\vec{F}_T = -\kappa\vec{r}\\
\vec{\nabla}\cdot\vec{F}_T = -3\kappa\\
\frac{6n\sigma_R\sigma_L I}{c} - \frac{6n\sigma_L^2 I}{c} - 3\kappa = 0\\
n_{max} = \frac{c\kappa}{2\sigma_L(\sigma_R - \sigma_L)I}
\end{gathered}$$

(d) In the limit of non-resonant excitation ($| \omega_L - \omega_0 | >> \omega_R, \Gamma$) and for a red laser detuning, the fluorescence spectrum should look something like figure a (see next page)
The laser emission spectrum is of course centered at ω_L (figure b),
and the atom's absorption profile is a single peak at ω_0 (figure c)

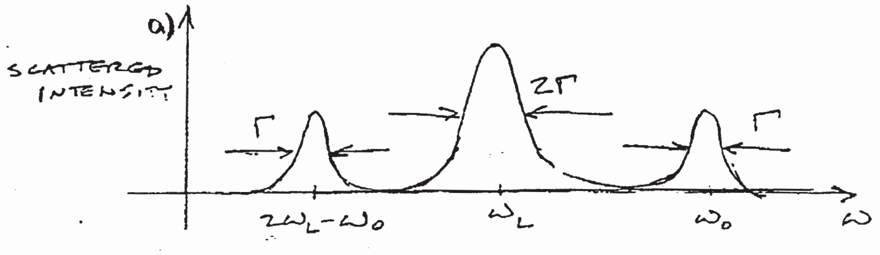
So we expect that the upper side-band of the fluorescence spectrum (i.e. the spectrum of the reradiated light) will overlap with the absorption profile significantly more than the laser spectrum does, and hence that $\sigma_R > \sigma_L$. This condition allows for a stable trap.

(e) Dark SPOT

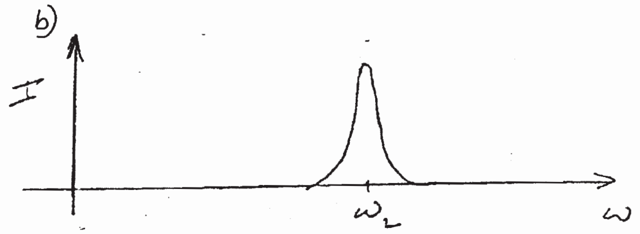
The trapping force depends on one scattering event, whereas the attenuation and radiation trapping forces depend on two scattering events. So if a fraction f of the atoms absorbs the trapping light, the forces vary with f as:

$$F_A, F_R \sim f^2$$
$$F_T \sim f.$$

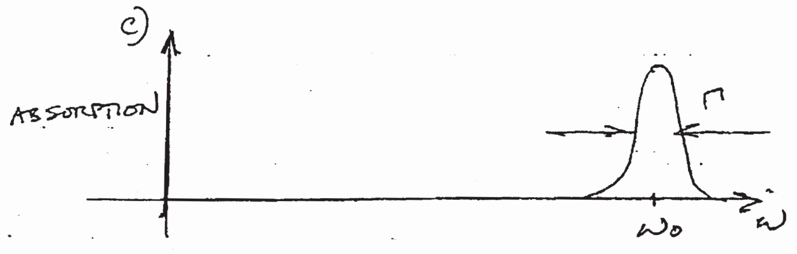
Thus $n_{max} \sim 1/f$, and using a ”dark” trapping scheme can increase the density dramatically. Typically, for Bose condensate production using magneto-optical traps, one needs as much density as possible in a MOT. Therefore, many BEC machinces use a dark-spot MOT as the first step in collecting atoms for evaporation. See PRL **70**, 2253 (1993) for more information.



FLUORESCENCE SPECTRUM



LASER SPECTRUM



ABSORPTION SPECTRUM

3. Optical Dipole Trap

a) The EM field at the position D is given by $\overrightarrow{E_D} \propto \hat{x}e^{i(wt-kz-\delta)}$ where δ is the phase shift due to the atoms in the reservoir B. Using $n_r \simeq 1 + \frac{1}{2}n(t)\alpha(w)$, the phase shift δ is

$$\begin{aligned}\delta &= \frac{\omega d}{c}(n_r - 1) \\ &= \frac{\omega d}{2c}n(t)\alpha(w).\end{aligned}$$

Therefore, it is clear that the phase shift is constant at $t < t_0$ and $t > t_0 + \Delta t$, which means that there is no frequency shift. However, when the atomic density n(t) is changing linearly, we can write the exponent of EM field as follows:

$$\omega t - kz - \delta = t[\omega - \frac{\omega d}{c}2\pi\dot{n}\alpha(w)] \text{ (because } n(t) = \dot{n}t \text{ when } t_0 < t < t_0 + \Delta t)$$

Thus, the frequency shift is

$$\Delta\omega = \frac{\omega d}{2c}\dot{n}\alpha(w)$$

b) The energy loss per photon (due to frequency shift) between time t_0 and $t_0 + \Delta t$ is given by

$$\Delta E_{photon} = \hbar\Delta\omega$$

Therefore,

$$\Delta E_{EM} = \hbar\Delta\omega \times N_{photon}$$

If we write down E-field as $\vec{E} = \hat{x}E_0e^{wt-kz}$,

$$N_{photon} = \frac{E_0^2 c\Delta t}{\hbar\omega}.$$

Therefore, the energy loss in EM field (due to frequency shift) is

$$\begin{aligned}\Delta E_{EM} &= \hbar\Delta\omega \times \frac{E_0^2 c\Delta t A}{\hbar\omega} \\ &= \frac{1}{2}E_0^2 A d\Delta t\dot{n}\alpha\end{aligned}$$

However, there is additional change in total energy of EM fields inside the reservoir B. At time $t > t_0 + \Delta t$, the induced dipole moments of atoms changes the total EM energy inside the reservoir B:

$$\begin{aligned}\Delta E_{inside}|_{t=t_o+\Delta t} &= \int_{Ad}[dV\frac{1}{2}\overrightarrow{D}\cdot\overrightarrow{E} - \frac{1}{2}\overrightarrow{E}\cdot\overrightarrow{E}] \\ &= \int_{Ad} dV[\frac{1}{2}\overrightarrow{P}\cdot\overrightarrow{E}] \\ &= \frac{1}{2}E_0^2 A d\Delta t\dot{n}\alpha\end{aligned}$$

Therefore, the total energy decrease of EM fields is

$$\begin{aligned}\Delta E_{totalEM} &= \Delta E_{EM} + \Delta E_{inside} \\ &= E_0^2 A d\Delta t\dot{n}\alpha\end{aligned}$$

c)Quadratic Stark effect gives us,

$$\Delta E_{Starkperatom} = \frac{E_0^2}{2}\alpha$$

Therefore, total energy gained through AC stark shift is given by

$$\begin{aligned}\Delta E_{Stark} = \Delta E_{Starkperatom} \times N_{atom} \\ = \frac{E_0^2}{2}\alpha \times (\dot{n}\Delta t)Ad \\ = \frac{1}{2}E_0^2 Ad\Delta t\dot{n}\alpha\end{aligned}$$

This result shows that total energy cost in EM fields is twice as large as the total Stark effect energy. This explains that a half of total energy cost in EM fields is required to induce dipole moment of the atom and the other half to accelerate the atom mechanically. The result shows how a far-off resonant (red-detuned) dipole trap accelerates atoms by stimulated light force.